THE PLANE PARTITION FUNCTION ABIDES BY BENFORD'S LAW

Katherine Douglass[1], Ken Ono[2]

In 2011, Anderson, Rolen and Stoehr proved the beautiful theorem that the partition function $p(n)$ abides by "Benford's Law", which means that

$$\lim_{X\to+\infty} \frac{\#\{0 \le n \le X \ :\ p(n) \text{ in base } b \text{ begins with string } f\}}{X} = \log_b(f+1) - \log_b(f) \pmod 1.$$

Here we prove that MacMahon's plane partition function $\mathrm{PL}(n)$ *also abides by Benford's Law. This result is obtained by applying their general method to strong asymptotics for* $\mathrm{PL}(n)$.

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1. Introduction

A *partition* of a non-negative integer n is any non-increasing sequence of positive integers that sums to n, and the *partition function* $p(n)$ counts their number. For example, the partitions of 5 are:

$$5,\quad 4+1,\quad 3+2,\quad 3+1+1,\quad 2+2+1,\quad 2+1+1+1,\quad 1+1+1+1+1,$$

and so we have that $p(5) = 7$. Ramanujan's celebrated congruences [18, 19] assert, for every non-negative integer n, that

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5,\\ p(7n+5) &\equiv 0 \pmod 7,\\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

These congruences have inspired many works (for example, see [1, 2, 4, 5, 6, 7, 10, 11, 13, 14, 16, 20] to name a few). These congruences can be thought of as arithmetic properties of the "insignificant" digits of $p(n)$ when written in base 5, 7, and 11. Indeed, Ramanujan's congruence modulo 5 is the assertion that every fifth partition number, beginning with $p(4)$, has units digit 0 when written in base 5.

It is natural to speculate on the arithmetic properties of the "significant digits". For example, in the usual base 10, what can be said about the distribution of the values of the first digit, an integer in $\{1, 2, 3, \ldots, 9\}$. A naive guess would be that the values of $p(n)$ begin with each of the nine possible values with equal likelihood (i.e. asympotically one ninth of the time). After all, why would one expect certain digits to be favored over others?

[1] Graduate Student, Department of Statistics, University of Virginia, Charlottesville, VA 22901, USA, E-mail: `kcd6znz@virginia.edu`

[2] Professor, Department of Mathematics, University of Virginia, Charlottesville, VA 22901, USA, E-mail: `ken.ono691@virginia.edu`

Minimal numerics immediately cast significant doubt on this naive guess. To make this precise, for an integer sequence $\{a(0), a(1), \ldots\}$ we define

$$B_a(f, b; X) := \frac{\#\{0 \leq n < X \ : \ a(n) \text{ in base } b \text{ begins with the string } f\}}{X}. \tag{1}$$

For base $b = 10$, Table 1 suggests that the naive guess is badly false, where for convenience we let $B_f(X) := B_p(f, 10; X)$.

TABLE 1. Distribution of the first base 10 digit of $p(n)$

X	$B_1(X)$	$B_2(X)$	$B_3(X)$	$B_4(X)$	$B_5(X)$	$B_6(X)$	$B_7(X)$	$B_8(X)$	$B_9(X)$
10^2	0.33	0.16	0.14	0.09	0.07	0.06	0.07	0.05	0.03
10^3	0.305	0.177	0.127	0.094	0.076	0.068	0.057	0.052	0.044
10^4	0.302	0.177	0.126	0.096	0.078	0.067	0.057	0.051	0.046
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

In a beautiful paper, Anderson, Rolen and Stoehr [3] determined the limiting distribution of arbitrary initial strings of the partition function in every base b. They proved (see Corollary 2 of [3]) that $p(n)$ satisfies *Benford's Law*[1] in every base b, which means that

$$\lim_{X \to +\infty} B_p(f, b; X) \equiv \log_b(f+1) - \log_b(f) \pmod 1. \tag{2}$$

This phenomenon is nicely illustrated by comparing Table 1 with the entries in Table 2, where we let $L_f := \log_{10}(f+1) - \log_{10}(f)$.

TABLE 2. Benford's Law for the first base 10 digit of $p(n)$

f	1	2	3	4	5	6	7	8	9
$L_{10}(f)$	0.301	0.176	0.125	0.097	0.079	0.067	0.058	0.051	0.046

Here we show that their result also holds for *plane partitions* (for background, see [4]). A *plane partition* of size n is an array of non-negative integers $\pi := (\pi_{i,j})$ for which $|\pi| := \sum_{i,j} \pi_{i,j} = n$, in which the rows and columns are weakly decreasing. The figure below offers a 3-dimensional rendering of a plane partition.

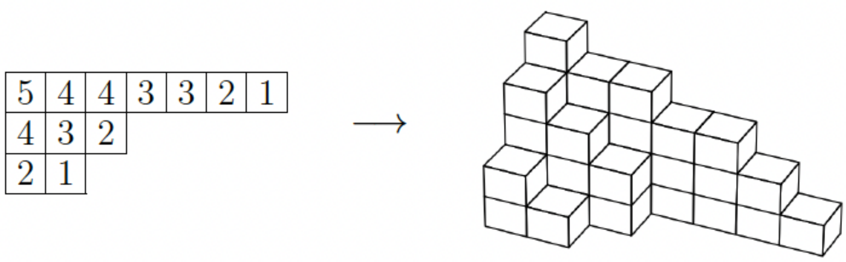

FIGURE 1. Example of a plane partition

[1]Benford famously observed [8] that certain data sets seem to empirically satisfy "logarithmic distributions."

If PL(n) is the number of size n plane partitions, then MacMahon [12] proved that

$$F(x) = \sum_{n=0}^{\infty} \mathrm{PL}(n)x^n := \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^n} = 1 + x + 3x^2 + 6x^3 + 13x^4 + 24x^5 + 48x^6 + \dots. \quad (3)$$

Remark 1.1. *Plane partitions appear prominently in physics in connection with the enumeration of small black holes in string theory. Indeed, $F(x)$ is the generating function (see Appendix E of* [9]*) for the number of BPS bound states between a $D6$ brane and $D0$ branes on $\mathbb{C}^3$.*

In analogy with the result of Anderson, Rolen and Stoehr for the partition function $p(n)$, we prove the following for MacMahon's plane partition function PL(n).

Theorem 1.1. *The plane partition function* PL(n) *abides by Benford's Law.*

Example 1.1. *Here we illustrate Theorem 1.1 in base 2 (i.e. binary), which means $n = d_m \cdot 2^m + d_{m-1} \cdot 2^{m-1} + \dots + d_1 \cdot 2^1 + d_0$, where each $d_j \in \{0_2, 1_2\}$. Every integer $n \geq 4$ has at least 3 digits in binary, and for them we can study the distribution of the initial 3 digits strings, which must be one of the four possibilities $f \in \{1_2 0_2 0_2, 1_2 0_2 1_2, 1_2 1_2 0_2, 1_2 1_2 1_2\} = \{4, 5, 6, 7\}$. Table 3 includes some numerical data,*

TABLE 3. Distribution of the initial 3 binary digits of PL(n)

X	$B_{\mathrm{PL}}(1_2 0_2 0_2, 2; X)$	$B_{\mathrm{PL}}(1_2 0_2 1_2, 2; X)$	$B_{\mathrm{PL}}(1_2 1_2 0_2, 2; X)$	$B_{\mathrm{PL}}(1_2 1_2 1_2, 2; X)$
200	0.345	0.244	0.269	0.142
400	0.333	0.255	0.252	0.161
600	0.327	0.255	0.246	0.173
800	0.326	0.255	0.237	0.178
1000	0.333	0.250	0.237	0.181
⋮	⋮	⋮	⋮	⋮
1500	0.331	0.251	0.235	0.184

Theorem 1.1 is nicely illustrated by comparing Table 3 with the entries in Table 4, where we let $L_2(f) := \log_2(f+1) - \log_2(f)$.

TABLE 4. Benford's Law for the initial 3 binary digits of PL(n)

f	4	5	6	7
$L_2(f)$	0.322	0.263	0.222	0.193

Theorem 1.1 implies that every possible string f occurs as the initial string for a positive proportion of n. Therefore, it is natural to ask for an effective upper bound for the first such n.

Problem. *For each string f in base b, determine a bound $N(f; b)$ with the property that there is a non-negative integer $n \leq N(f, b)$ for which* PL(n) *begins with string f in base b.*

To prove Theorem 1.1, we make use of a general criterion for "Benfordness" that was obtained by Anderson, Rolen and Stoehr [3] (see Section 2.**2**). This criterion requires strong information about the asymptotics of PL(n). Namely, we require precise limiting statements regarding continuous functions that enjoy the same asymptotics as PL(n). To this end, we give strong asymptotics for PL(n) in Section 2.**1**. Using these asymptotics, we are able to employ this criterion to prove Theorem 1.1 in Section 2.**3**.

2. The proof

2.1. Asymptotics for the plane partition function

The key device for proving that $p(n)$ abides by Benford's Law is the generating function

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + \dots. \tag{4}$$

As is well known, this generating function is essentially given by the weight -1/2 modular form

$$\frac{1}{\eta(z)} = \frac{q^{-\frac{1}{24}}}{\prod_{n=1}^{\infty}(1-q^n)} = \sum_{n=0}^{\infty} p(n)q^{n-\frac{1}{24}},$$

where $q := e^{2\pi i z}$ and $\eta(z)$ is Dededkind's eta-function (for example, see [15]). Anderson, Rolen and Stoehr prove the Benfordness of $p(n)$ using the Hardy-Ramanujan asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{2n/3}},$$

that is derived from the modularity above.

Similarly, to prove Theorem 1.1, we require a strong asymptotic formula for $\mathrm{PL}(n)$. In the 1930s, Wright [21] adapted the "circle method" of Hardy and Ramanujan to prove asymptotic formulas for $\mathrm{PL}(n)$. He obtained such a formula for every positive integer r, where the implied error terms are smaller with larger choices of r for large n.

In recent work, Pujahari, Rolen and the second author [17] made this result effective and explicit. Namely, they explicitly bounded the error in the approximation for each r. To state these formulas, we require the constants

$$A := \zeta(3) \approx 1.202056\dots \quad \text{and} \quad c := 2\int_0^{\infty} \frac{y \log y}{e^{2\pi y}-1}dy = \zeta'(-1) \approx -0.16542\dots. \tag{5}$$

Furthermore, for any pair of non-negative integers s and m, we define coefficients $c_{s,m}(n)$ by $\frac{(1+y)^{2s+2m+\frac{13}{12}}}{(3+2y)^{(m+\frac{1}{2})}} =: \sum_{n=0}^{\infty} c_{s,m}(n)y^n$. In terms of these coefficients, we define the important numbers

$$b_{s,m} := c_{s,m}(2m). \tag{6}$$

The asymptotic formulas are defined in terms of special numbers $\beta_0, \beta_1, \dots$. To define them, for every positive integer s we let

$$\alpha_s := \frac{2\Gamma(2s+2)\zeta(2s)\zeta(2s+2)}{s(2\pi)^{4s+2}}, \tag{7}$$

where $\zeta(s)$ is Riemann's zeta-function. The real numbers β_s are the Taylor coefficients of

$$\exp\left(-\sum_{i=1}^{\infty} \alpha_i y^i\right) =: \sum_{n=0}^{\infty} \beta_s y^s. \tag{8}$$

For positive integers r, and using the numbers $\beta_0, \dots, \beta_{r+1}$, we recall the following recent explicit asymptotic formulae due to Pujahari, Rolen and the second author.

Theorem 2.1 (Thm. 1.3 of [17]). *If $r \in \mathbb{Z}^+$, then for every integer $n \geq \max(n_r, \ell_r, 87)$ (see (2.8) and (2.9) of* [17]*) we have*

$$\mathrm{PL}(n) = \frac{e^{c+3AN_n^2}}{2\pi} \sum_{s=0}^{r+1}\sum_{m=0}^{r+1} \frac{(-1)^m \beta_s b_{s,m}\Gamma\left(m+\frac{1}{2}\right)}{A^{m+\frac{1}{2}}N_n^{2s+2m+\frac{25}{12}}} + E_r^{\mathrm{maj}}(n) + E^{\mathrm{min}}(n),$$

where $|E_r^{\mathrm{maj}}(n)| \leq \widehat{E}_r^{\mathrm{maj}}(n)$ *(see definition (2.36) of* [17]*),* $N_n := (\frac{n}{2A})^{\frac{1}{3}}$ *and*

$$|E^{\min}(n)| \leq \exp\left(\left(3A - \frac{2}{5}\right) n^2/(2A)^{\frac{2}{3}}\right).$$

2.2. Work by Anderson, Rolen and Stoehr

Anderson, Rolen and Stoehr determined analytical properties for integer sequences that guarantee Benfordness. Namely, they made the following crucial definition.

Definition 2.1. *We say that an integer sequence* $\{a(0), a(1), \dots\}$ *is good whenever* $a(n) \sim b(n)e^{c(n)}$ *(where* $f(x) \sim g(x)$ *means that* $\lim_{x\to+\infty} f(x)/g(x) = 1$*) and the following conditions are satisfied:*

(1) There exists some integer $h \geq 1$ *such that* $c(n)$ *is* h*-differentiable and* $c^{(h)}(n)$ *tends to zero monotonically for sufficiently large* n*.*

(2) We have that

$$\lim_{n\to+\infty} n|c^{(h)}(n)| = +\infty.$$

(3) We have that

$$\lim_{n\to+\infty} \frac{D^{(h)}(\log b(n))}{c^{(h)}(n)} = 0,$$

where $D^{(h)}$ *denotes the* h*th derivative.*

The following theorem is their main technical result in [3].

Theorem 2.2 (Thm. 1.1 of [3])**.** *Good integer sequences abide by Benford's Law.*

2.3. Proof of Theorem 1.1

We use Theorem 2.2 to prove Theorem 1.1. Although PL(n) is not a continuous and differentiable function in n, we can make use of Theorem 2.1 to replace it by a differentiable function in n that has the same initial strings for all sufficiently large n, as they satisfy the same asymptotics.

To this end, we let $s = m = 0$ in Theorem 2.1, and we obtain the asymptotic

$$\mathrm{PL}(n) \sim \frac{(2^{25}A^7)^{\frac{1}{36}} e^c}{\sqrt{12\pi} \cdot n^{\frac{25}{36}}} \exp\left(\sqrt[3]{\frac{27An^2}{4}}\right).$$

In the notation of Theorem 2.2, we have that

$$\mathrm{PL}(n) \sim b(n)e^{c(n)},$$

where we have

$$b(n) := \frac{(2^{25}A^7)^{\frac{1}{36}} e^c}{\sqrt{12\pi} \cdot n^{\frac{25}{36}}} = \beta \cdot n^{-\frac{25}{36}} \quad \text{and} \quad c(n) := \sqrt[3]{\frac{27An^2}{4}} = \gamma n^{\frac{2}{3}}, \tag{9}$$

where $\beta, \gamma > 0$. Now we check the conditions for "goodness" one-by-one with $h = 1$.

We find that

$$c'(n) = \frac{2\gamma}{3\sqrt[3]{n}}$$

Obviously, we have that $1/\sqrt[3]{n} \to 0$ monotonically as $n \to \infty$, confirming the first condition in the definition. Similarly, one directly finds that

$$\lim_{n\to+\infty} n|c'(n)| = \lim_{n\to+\infty} \frac{2\gamma n^{\frac{2}{3}}}{3} = +\infty.$$

The second condition follows as $n^{\frac{2}{3}} \to +\infty$ as $n \to +\infty$. Finally, we check the third and final condition as follows

$$\lim_{n\to+\infty} \frac{\frac{d}{dn}(\log b(n))}{c'(n)} = -\lim_{n\to+\infty} \frac{25}{24\gamma n^{\frac{2}{3}}} = 0,$$

confirming the third condition, and the fact that PL(n) is good. Therefore, Theorem 2.2 implies that PL(n) abides by Benford's Law, completing the proof of the theorem.

REFERENCES

[1] *S. Ahlgren*, The partition function modulo composite integers M, Math. Ann. **318** (2000), pages 793-803.

[2] *S. Ahlgren and K. Ono*, Congruence properties for the partition function, Proc. Natl. Acad. Sci., USA **98** (2001), pages 12882-12884.

[3] *T. Anderson, L. Rolen and R. Stoehr*, Benford's law for coefficients of modular forms and partition functions, Proceedings of the American Mathematical Society **139** (2011), 1533-1541.

[4] *G. E. Andrews*, The theory of partitions, Cambridge Univ. Press, Cambridge, 1998.

[5] *A. O. L. Atkin*, Proof of a conjecture of Ramanujan, Glasgow Math. J. **8** (1967), pages 14-32.

[6] *A. O. L. Atkin*, Multiplicative congruence properties and density problems for $p(n)$, Proc. London Math. Soc. (3) **18** (1968), pages 563-576.

[7] *A. O. L. Atkin and J. N. O'Brien*, Some properties of $p(n)$ and $c(n)$ modulo powers of 13, Trans. Amer. Math. Soc. **126** (1967), pages 442-459.

[8] *F. Benford*, The law of anomalous numbers, Proceedings of the American Philosophical Society **78** (1938), 551-572.

[9] *A. Dabholkar, F. Denef, G. Moore, and B. Pioline*, Precision counting of small black holes, J. High Energy Physics (2005), Art. 096.

[10] *M. Hirschhorn and D.C. Hunt,* A simple proof of the Ramanujan conjecture for powers of 5, J. Reine Angew. Math. **326** (1981), pages 1–17.

[11] *J. Lovejoy and K. Ono*, Extension of Ramanujan's congruences for the partition function modulo powers of 5, J. Reine Ange. Math. **542** (2002), pages 123-132.

[12] *P. A. MacMahon*, Combinatory analysis, Vol. I, II, Dover Publ., Mineola, New York, 2004.

[13] *M. Newman*, Periodicity modulo m and divisibility properties of the partition function, Trans. Amer. Math. Soc. **97** (1960), pages 225-236.

[14] *M. Newman*, Congruences for the coefficients of modular forms and some new congruences for the partition function, Canadian J. Math. **9** (1957), pages 549-552.

[15] *K. Ono*, The web of modularity: arithemtic of the coefficient of modular forms and q-series. CBMS Regional Conference Series in Mathematics, **102**, Amer. Math. Soc., Providence, RI 2004.

[16] *K. Ono,* The partition function modulo m, Ann. of Math. **151** (2000), pages 293-307.

[17] *K. Ono, S. Pujahari, and L. Rolen*, Turán inequalities for the plane partition function, Advances in Mathematics **409** (2022), Paper No. 108692.

[18] *S. Ramanujan*, Congruence properties of partitions, Proc. London Math. Soc. (2) **19**, pages 207-210.

[19] *S. Ramanujan,* Ramanujan's unpublished manuscript on the partition and tau-functions (Commentary by B. C. Berndt and K. Ono) The Andrews Festschrift (Ed. D. Foata and G.-N. Han), Springer-Verlag, Berlin, 2001, pp. 39-110.

[20] *G. N. Watson*, Ramanujan's vermutung über zerfällungsanzahlen, J. Reine Angew. Math. **179** (1938), pages 97-128.

[21] *E. M. Wright*, Asymptotic partition formulae, I: Plane partitions, Quarterly J. Math. Oxford, **2**, (1931), 177-189.