SPECIAL ELEMENTS AND PSEUDO INVERSE FUNCTIONS IN GROUPOIDS

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In this paper, we consider a theory of elements u of a groupoid $(X, *)$ which have associated with them certain functions $\widehat{u} : X \to X$, pseudo-inverse functions, which are generalizations of the inverses associated with units of groupoids with identity elements. It turns out that if we classify the elements u as special of one of twelve types, then it is possible to do a rather detailed analysis of certain cases, leftoids, rightoids and linear groupoids included, which demonstrates that it is possible to develop a successful theory and that a good deal of information has already been obtained with much more possible in the future.

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1. Introduction

R. H. Bruck ([5]) published a book, *A survey of binary systems* discussed in the theory of groupoids, loops and quasigroups, and several algebraic structures. O. Borůvka ([4]) stated the theory of decompositions of sets and its application to binary systems. Recently, some interesting results in groupoids were investigated by several researchers [3, 11, 13, 19]. Semigroups are in fact the first and simplest type of algebra to which the methods of universal algebra must be applied, and any mathematician interested in universal algebra will find semigroup theory a rewarding study ([10]). The notion of d-algebras which is another useful generalization of BCK-algebras was introduced by J. Neggers and H. S. Kim ([17]), and some relations between d-algebras and BCK-algebras as well as several other relations between d-algebras and oriented digraphs were investigated. Several aspects on d-algebras were studied ([1, 9, 12, 15, 16]). H. S. Kim and J. Neggers [8] introduced the notion of $Bin(X)$ of all binary systems(groupoids, algebras) defined on a set X, and showed that it becomes a semigroup under suitable operation.

In this paper, we consider a theory of elements u of a groupoid $(X, *)$ which have associated with them certain functions $\widehat{u} : X \to X$, pseudo-inverse functions, which are generalizations of the inverses associated with units of groupoids with identity elements.

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It turns out that if we classify the elements u as special of one of twelve types, then it is possible to do a rather detailed analysis of certain cases, leftoids, rightoids and linear groupoids included, which demonstrates that it is possible to develop a successful theory and that a good deal of information has already been obtained with much more possible in the future.

2. Preliminaries

A *d-algebra* ([17]) is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms: (I) $x*x=0$, (II) $0*x=0$, (III) $x*y=0$ and $y*x=0$ imply $x=y$ for all $x,y\in X$. For brevity we also call X a *d-algebra*. In X we can define a binary relation "$\leq$" by $x\leq y$ if and only if $x*y=0$. For general references on d-algebras we refer to [1, 2, 9, 12, 15, 16]. A BCK-algebra is a d-algebra X satisfying the following additional axioms: (IV) $(x*y)*(x*z))*(z*y)=0$, (V) $(x*(x*y))*y=0$ for all $x,y,z\in X$.

Given two groupoids $(X,*)$ and $(X,\bullet)$, we define a new binary operation $\square$ by $x\square y:=(x*y)\bullet(y*x)$ for all $x,y\in X$. Then we obtain a new groupoid $(X,\square)$, i.e., $(X,\square)=(X,*)\square(X,\bullet)$. We denote the collection of all binary systems(groupoid, algebras) defined on X by $Bin(X)$ [8].

Theorem 2.1. [8] *$(Bin(X),\square)$ is a semigroup and the left zero semigroup is an identity.*

3. LL-special and pseudo inverse functions

Given a groupoid $(X,*)$, i.e., $(X,*)\in Bin(X)$, an element $u\in X$ is said to be *LL-special* if there exists a map $\widehat{u}:X\to X$ such that $(\widehat{u}(x)*u)*x=x$ for all $x\in X$. Such a function $\widehat{u}$ is said to be a *pseudo inverse function* of u and $\widehat{u}(x)$ is called a *pseudo inverse* of u with respect to x.

Example 3.1. Let $(X,*)$ be a right-zero semigroup. For any $u\in X$, for any $\widehat{u}\in X^X$, we have $(\widehat{u}(x)*u)*x=u*x=x$ for all $x\in X$. This shows that every element u of a right-zero semigroup is LL-special and every function $\widehat{u}:X\to X$ is a pseudo inverse function of u.

Example 3.1 shows that a pseudo inverse function $\widehat{u}$ need not be unique.

Example 3.2. Let $(X,*)$ be a left-zero semigroup and let u be an LL-special element of X. Then $(\widehat{u}(x)*u)*x=x$ for all $x\in X$. It follows that $\widehat{u}(x)=x$ for all $x\in X$, which means that a pseudo inverse function $\widehat{u}$ of u is the identity map on X.

Proposition 3.3. *Let $(X,*,e)$ be a group. Then*

(i) *every element of X is LL-special,*
(ii) *if u is LL-special, then its pseudo inverse function $\widehat{u}$ is a constant map.*

Proof. (i) Given $u\in X$, we define a map $\widehat{u}:X\to X$ by $\widehat{u}(x):=u^{-1}$ for all $x\in X$. Then $(\widehat{u}(x)*u)*x=(u^{-1}*u)*x=e*x=x$, which shows that u is LL-special. (ii) Assume u is LL-special. Then there exists a map $\widehat{u}:X\to X$ such that $(\widehat{u}(x)*u)*x=x$ for all $x\in X$. Since $(X,*)$ is a group, we have $\widehat{u}(x)*u=e$, and hence $\widehat{u}(x)=u^{-1}$ for all $x\in X$, which proves that $\widehat{u}$ is a constant map. $\square$

Remark 3.4. Given $(X,*) \in Bin(X)$, a condition $X = X*X$ is a necessary condition for the existence of LL-special elements of $(X,*)$. In fact, if u is LL-special in $(X,*)$, then there exists a map $\widehat{u} : X \to X$ such that $(\widehat{u}(x) * u) * x = x$ for all $x \in X$. If we let $\alpha(x) := \widehat{u}(x) * u$, then $\alpha(x) * x = x$ for all $x \in X$, which shows that $X \subseteq X * X$.

Proposition 3.5. *Any leftoid $(X,*)$, where $x*y := f(x)$ for all $x, y \in X$, $f : X \to X$ is not onto, is a groupoid which does not contain any LL-special element of $(X,*)$.*

Proof. Since $f : X \to X$ is not onto, there exists $x_0 \in X$ such that $x_0 \notin f(X)$. Assume u is an LL-special element of X. Then there exists a map $\widehat{u} : X \to X$ such that $(\widehat{u}(x) * u) * x = x$ for all $x \in X$. It follow that $(\widehat{u}(x_0) * u) * x_0 = x_0$. If we let $\alpha := \widehat{u}(x_0) * u$, then $x_0 = \alpha * x_0 = f(\alpha) \in f(X)$, since $(X,*)$ is a leftoid, which leads to a contradiction. This proves the proposition. □

Proposition 3.6. *Let $(X,*)$ be a leftoid for f, i.e., $x * y := f(x)$ for all $x, y \in X$. Let $u \in X$ and $\widehat{u} : X \to X$ be a map. Then u is LL-special if and only if $f^2(\widehat{u}(x)) = x$ for all $x \in X$.*

Proof. It follows immediately from $x = (\widehat{u}(x) * u) * x = f(\widehat{u}(x)) * x = f(f(\widehat{u}(x)))$. for all $x \in X$. □

Example 3.7. Let $X := \mathbf{R}$ be the set of all real numbers. We define $x * y := x^2$, i.e., $x * y = f(x) = x^2$ for all $x, y \in \mathbf{R}$. Assume u is LL-special. Then, by Proposition 3.6, we have $x = (\widehat{u}(x) * u) * x = f^2(\widehat{u}(x)) = (\widehat{u}(x))^4$ for all $x \in \mathbf{R}$. If we let $x := -1$, then $(\widehat{u}(-1))^4 = -1$, a contradiction.

Theorem 3.8. *Let $(X,*)$ be a semigroup and let u, v be LL-special elements of $(X,*)$. Then $(\widehat{u}(x) * u) * \widehat{v}(x)$ is another pseudo inverse of v with respect to x for all $x \in X$.*

Proof. Assume that u, v are LL-special elements of $(X,*)$. Then there exist pseudo inverse functions $\widehat{u}, \widehat{v}$ respectively. For any $x \in X$, we have $x = (\widehat{u}(x) * u) * x = (\widehat{u}(x) * u) * [(\widehat{v}(x) * v) * x)] = [(\widehat{u}(x) * u) * (\widehat{v}(x) * v)] * x = [\{(\widehat{u}(x) * u) * \widehat{v}(x)\} * v] * x$. If we take a map $\widehat{v}'(x) := (\widehat{u}(x) * u) * \widehat{v}(x)$ for all $x \in X$, then $\widehat{v}, \widehat{v}'$ are pseudo-inverse functions of v with respect to x. □

4. Several special elements

Let $(X,*) \in Bin(X)$. An element u of X is said to be

- *LL-special* : $(\widehat{u}(x) * u) * x = x$, • *$RL'$-special* : $(x * \widehat{u}(x)) * u = x$,
- *LR-special* : $\widehat{u}(x) * (u * x) = x$, • *$RR'$-special* : $x * (\widehat{u}(x) * u) = x$,
- *RL-special* : $u * (\widehat{u}(x) * x) = x$, • *$LL''$-special* : $(\widehat{u}(x) * x) * u = x$,
- *RR-special* : $(u * \widehat{u}(x)) * x = x$, • *$LR''$-special* : $\widehat{u}(x) * (x * u) = x$,
- *LL'-special* : $x * (u * \widehat{u}(x)) = x$, • *$RL''$-special* : $u * (x * \widehat{u}(x)) = x$,
- *LR'-special* : $(x * u) * \widehat{u}(x) = x$, • *$RR''$-special* : $(u * x) * \widehat{u}(x) = x$

for all $x \in X$, where $\widehat{u} : X \to X$ is a map.

We note that if $(X,*)$ is a semigroup, then $(x * y) * z = x * (y * z)$ and thus we find that $(\widehat{u}(x) * u) * x = \widehat{u}(x) * (u * x)$, and "$LL$-special = LR-special". Similarly, we find that "RL-special = RR-special", "LR'-special = LL'-special", "RL'-special = RR'-special", "LL''-special = LR''-special" and "RL''-special = RR''-special". Next, suppose that $(X,*)$ is a commutative groupoid, i.e., $x * y = y * x$ for all $x, y \in X$. Then $(\widehat{u}(x) * u) * x =$

$(u * \widehat{u}(x)) * x = x * (u * \widehat{u}(x)) = x * (\widehat{u}(x) * u)$, and hence "$LL$-special = RR-special = LL'-special = RR'-special". Since $\widehat{u}(x) * (u * x) = \widehat{u}(x) * (x * u) = (x * u) * \widehat{u}(x)) = (u * x) * \widehat{u}(x)$, we have "$LR$-special = LR''-special = LR'-special = RR''-special". Moreover, since $u * (\widehat{u}(x) * x) = u * (x * \widehat{u}(x)) = (x * \widehat{u}(x)) * u = (\widehat{u}(x) * x) * u$, we have "$RL$-special = RL''-special = RL'-special = LL''-special". If $(X, *)$ is a commutative semigroup, then all 12 types of special elements into a single type. In this case, we call u a *special element* of $(X, *)$.

Example 4.1. Let $X := [0, \infty)$ and let $x * y := x + y$ for all $x, y \in X$ where "+" is the usual addition in real numbers. Assume $\widehat{u}(x) + u + x = x$. Then $\widehat{u}(x) + u = 0$ for all $x \in X$. It follows that $\widehat{u}(x) = u = 0$ for all $x \in X$. Hence $u = 0$ is the only special element of $(X, *)$ and $\widehat{u}(x) = 0$ for all $x \in X$, i.e., the zero map on X.

Example 4.2. Let $X := (0, \infty)$ and let $x * y := x + y$ for all $x, y \in X$ where "+" is the usual addition in real numbers. Assume $\widehat{u}(x) + u + x = x$. Then $\widehat{u}(x) + u = 0$ for all $x \in X$. It follows that $\widehat{u}(x) = u = 0 \notin X$. Hence $(X, *)$ has no special elements whatsoever.

Proposition 4.3. *If $(X, *, 0)$ is a BCK-algebra, then "LL-special = LL''-special", "RR-special = RR''-special" and "LL'-special = RL'-special".*

Proof. If $(X, *, 0)$ is a BCK-algebra, then $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$, and the proposition can be proved. □

Theorem 4.4. *Let K be a field and let $\alpha, \beta, \gamma \in K$. Define $x * y := \alpha + \beta x + \gamma y$ for all $x, y \in K$. If $(x * y) * z = (x * z) * y$ for all $x, y, z \in K$, then either $x * y = \alpha + \beta x$ or $x * y = \alpha + x + \gamma y$ for all $x, y \in K$.*

Proof. Given $x, y, z \in K$, we have $(x*y)*z = \alpha+\beta(x*y)+\gamma z = \alpha+\beta(\alpha+\beta x+\gamma y)+\gamma z = \alpha(1+\beta)+\beta^2 x+\beta\gamma y+\gamma z$. Similarly, we obtain $(x*z)*y = \alpha(1+\beta)+\beta^2 x+\beta\gamma z+\gamma y$. It follows that $0 = (x * y) * z - (x * z) * y = \beta\gamma(y - z) + \gamma(z - y) = \gamma(\beta - 1)(y - z)$ for all $y, z \in K$. This implies that either $\gamma = 0$ or $\beta = 1$, proving the theorem. □

Corollary 4.5. *Let K be a field and let $\alpha, \beta, \gamma \in K$. Define $x * y := \alpha + \beta x + \gamma y$ for all $x, y \in K$. If $(x * y) * z = (x * z) * y$ and $x * y = y * x$ for all $x, y, z \in K$, then either $x * y = \alpha$ or $x * y = \alpha + x + y$ for all $x, y \in K$.*

Proof. In the case of $x*y = \alpha+\beta x$, if $x*y = y*x$ for all $x, y \in K$, then $\alpha+\beta x = \alpha+\beta y$ and hence $\beta(x - y) = 0$ for all $x, y \in K$. This shows that $\beta = 0$, proving that $x * y = \alpha$. In the case of $x * y = \alpha + x + \gamma y$, if $x * y = y * x$ for all $x, y \in K$, then $\alpha + x + \gamma y = \alpha + y + \gamma x$ and hence $(1 - \gamma)(x - y) = 0$ for all $x, y \in K$. This shows that $\gamma = 1$, proving that $x * y = \alpha + x + y$. □

Proposition 4.6. *If $(X, *, 0)$ is a groupoid satisfying the conditions:*

(i) $x * y = y * x$ *for all* $x, y \in X$,
(ii) $(x * y) * z = (x * z) * y$ *for all* $x, y, z \in X$,

then it is a commutative semigroup and has a unique special element in $(X, *)$ if it exists.

Proof. Given $x, y, z \in X$, we have $(x * y) * z = (x * z) * y = (z * x) * y = (z * y) * x = x * (z * y) = x * (y * z)$, proving that $(X, *)$ is a commutative semigroup. Assume u_1, u_2 are special elements of $(X, *)$. Let $\widehat{u_i}(x) := \alpha (\in X)$ for all $x \in X$ $(i = 1, 2)$. Then $u_1 = (\widehat{u_1}(u_1) * u_2) * u_1$ and $u_2 = (\widehat{u_2}(u_2) * u_1) * u_2$. It follows that $u_1 = (\widehat{u_1}(u_1) * u_2) * u_1 = (\alpha * u_2) * u_1 = (\alpha * u_1) * u_2 = (\widehat{u_2}(u_2) * u_1) * u_2 = u_2$, proving the proposition. □

Corollary 4.7. *Let K be a field and let $\alpha \in K$. A groupoid $(K, *)$, where $x * y := \alpha$ or $x * y := \alpha + x + y$ for all $x, y \in K$, has a special element if it exists.*

Proof. It follows from Corollary 4.5 and Proposition 4.6. □

Proposition 4.8. *Let K be a field and let $\alpha, \beta, \gamma \in K$. Define $x * y := \alpha + \beta x + \gamma y$ for all $x, y \in K$. If $(K, *)$ is a semigroup, then $x * y = \alpha$, $x * y = \alpha + x$, $x * y = \alpha + y$ or $x * y = \alpha + x + y$ for all $x, y \in K$.*

Proof. It was proved that $(x * y) * z = \alpha(1 + \beta) + \beta^2 x + \beta\gamma y + \gamma z$. Similarly we obtain $x * (y * z) = \alpha(1 + \gamma) + \beta x + \gamma\beta y + \gamma^2 z$. It follows that $\alpha(1 + \beta) = \alpha(1 + \gamma), \beta^2 = \beta$ and $\gamma^2 = \gamma$, which proves the proposition. □

Remark. The property $(x * y) * z = (x * z) * y$ also holds for BCK-algebras and hence there is no guarantee in general for either commutativity or associativity as is the setting of the corollaries of the theorem.

5. Universally completely special elements

A groupoid $(X, *)$ is said to be *completely LL-special* if u is LL-special for all $u \in X$. A groupoid $(X, *)$ is said to be *universally completely special* if $(X, *)$ is completely α-special for any $\alpha \in \{LL, LR, RL, RR, LL', LR', RL', RR', LL'', LR'', RL'', RR''\}$.

Example 5.1. Define a binary operation "$*$" by $x * y := x^2$ for all $x, y \in X := [0, \infty)$, i.e., $(X, *)$ is a leftoid for $f(x) = x^2$. If u is LL-special in X, then $x = (\widehat{u}(x) * u) * x = f^2(\widehat{u}(x))$ for all $x \in X$. It follows that $\widehat{u}(x) = \sqrt[4]{x}$ for all $x \in X$. Hence $(X, *)$ is completely LL-special.

Example 5.2. Let $X := \mathbf{R}$, the set of all real numbers. Define a binary operation "$*$" on X by $x * y := \frac{1}{2}(x + y)$ for all $x, y \in X$. Given any $u \in X$, if we define $\widehat{u}(x) := 2x - u$ for all $x \in X$, then $(\widehat{u}(x) * u) * x = \frac{1}{2}[\frac{1}{2}(2x - u + u) + x] = x$, proving that u is LL-special. Hence $(X, *)$ is completely LL-special.

Note that every abelian group is universally completely special. Let $(X, *, e)$ be an abelian group. Then all 12 types of special elements coincide into a single type. Given $u \in X$, we let $(\widehat{u}(x) * u) * x = x$ for all $x \in X$. It follows that $\widehat{u}(x) * u = e$, i.e., $\widehat{u}(x) = u^{-1}$ for all $x \in X$. Hence u is completely LL-special for all $u \in X$.

Proposition 5.3. *Let $(X, *)$ be a leftoid for f, i.e., $x * y := f(x)$ for all $x, y \in X$. Let $u \in X$ and $\widehat{u} : X \to X$ be a map. Then u is LR-special if and only if $f(\widehat{u}(x)) = x$ for all $x \in X$.*

Proof. It follows immediately from $x = \widehat{u}(x) * (u * x) = f(\widehat{u}(x))$ for all $x \in X$. □

Example 5.4. Let $X := \mathbf{R}$ be the set of all real numbers. We define $x * y := x^3$, i.e., $x * y = f(x) = x^3$ for all $x, y \in \mathbf{R}$. Assume u is LR-special. Then, by Proposition 5.3, we have $x = f(\widehat{u}(x)) = (\widehat{u}(x))^3$ for all $x \in \mathbf{R}$. Hence $\widehat{u}(x) = \sqrt[3]{x}$ for all $x \in X$. This shows that $(\mathbf{R}, *, 0)$ is completely LR-special.

Proposition 5.5. *Let $(X, *)$ be a leftoid for f, i.e., $x * y := f(x)$ for all $x, y \in X$. Let $u \in X$ and $\widehat{u} : X \to X$ be a map. Then u is LR'-special if and only if $f^2(x) = x$ for all $x \in X$.*

Proof. It follows immediately from $x = (x * u) * \widehat{u}(x) = f(x) * \widehat{u}(x) = f(f(x))$ for all $x \in X$. □

Example 5.6. Let $(X, \cdot)$ be a group. We define a binary operation "$*$" on X by $x * y := x^{-1}$, i.e., $f(x) = x^{-1}$, for all $x, y \in X$. It follows that $f(f(x)) = (x^{-1})^{-1} = x$ for any $x \in X$, showing that $(X, *)$ is a leftoid for f and it is completely LR'-special.

Let K be a field and let $\alpha, \beta, \gamma \in K$. A groupoid $(K, *)$ is said to be a *linear groupoid* if $x * y := \alpha + \beta x + \gamma y$ for all $x, y \in K$. If $\beta = 0$, then $x * y = \alpha + \gamma y$ determines a rightoid, and if $\gamma = 0$, then $x * y = \alpha + \beta x$ determines a leftoid. Let $(X, *)$ be a groupoid. A mapping $\widehat{u} : X \to X$ is said to have a *fixed point* $x \in X$ if $\widehat{u}(x) = x$.

Theorem 5.7. *Let $(K, *)$ be a linear groupoid, i.e., $x * y := \beta x + \gamma y$ for all $x, y \in K$ where $\beta, \gamma \in K$. Let u be an LL-special element of $(K, *)$ and $\widehat{u}$ be a pseudo inverse of u. If $\widehat{u}$ has a fixed point of u, then either*

(i) *$x * y = -x + \gamma y$, $\gamma \neq 0$ and $\widehat{u}(x) = (1 - \gamma)x + \gamma u$, for all $x, y \in K$, or*

(ii) *$x * y = \beta x + (1 - \beta)y$, $\beta \neq 0$ and $\widehat{u}(x) = \frac{1}{\beta}(x - (1 - \beta)u)$, for all $x, y \in K$.*

Proof. Let $(K, *)$ be a linear groupoid, i.e., $x * y := \beta x + \gamma y$ for all $x, y \in K$ where $\beta, \gamma \in K$. Let u be an LL-special element of $(K, *)$. Then $x = (\widehat{u}(x) * u) * x$ for all $x \in K$. It follows that $x = \beta(\widehat{u}(x) * u) + \gamma x = \beta(\beta\widehat{u}(x) + \gamma u) + \gamma x = \beta^2\widehat{u}(x) + \beta\gamma u + \gamma x$. This shows that

$$\widehat{u}(x) = \frac{1 - \gamma}{\beta^2}x - \frac{u\beta\gamma}{\beta^2} \tag{1}$$

If we let $x := u$ in (1), then

$$\widehat{u}(u) = \frac{1}{\beta^2}[1 - \gamma(1 + \beta)]u \tag{2}$$

Since u is a fixed point of $\widehat{u}$, i.e., $\widehat{u}(u) = u$, we have $1 - \gamma(1 + \beta) = \beta^2$. It follows that $\gamma(1 + \beta) = (1 - \beta)(1 + \beta)$, and hence we have either $\beta = -1$ or $\gamma = 1 - \beta$. If $\beta = -1$, i.e., $x * y = -x + \gamma y, \gamma \neq 0$, then (1) leads to $\widehat{u}(x) = (1 - \gamma)x + \gamma u$ for all $x \in K$, which is the case (i). If $\gamma = 1 - \beta$, i.e., $x * y = \beta x + (1 - \beta)y, \beta \neq 0$, then (1) leads to $\widehat{u}(x) = \frac{1}{\beta}x - \frac{1-\beta}{\beta}u$, which is the case (ii). This completes the proof. □

Proposition 5.8. *Let $(K, *)$ be a linear groupoid with $x * y := \beta x + \gamma y$ for all $x, y \in K$ where $\beta \neq 0, \gamma \in K$. Let u be an LR-special element of $(K, *)$ and $\widehat{u}$ be a pseudo inverse of u. If $\widehat{u}$ has a fixed point of u, then $x * y = \beta x + (1 - \beta)y$ for all $x, y \in K$ and $\widehat{u}(x) = (2 - \beta)x + (\beta - 1)u$ for all $x \in K$.*

Proof. Let $(K, *)$ be a linear groupoid with $x * y := \beta x + \gamma y$ for all $x, y \in K$ where $\beta \neq 0, \gamma \in K$. Let u be an LR-special element of $(K, *)$. Then $x = \widehat{u}(x) * (u * x)$ for all $x \in K$. It follows that $x = \beta\widehat{u}(x) + \gamma(u * x) = \beta\widehat{u}(x) + \gamma(\beta u + \gamma x) = \beta\widehat{u}(x) + \beta\gamma u + \gamma^2 x$. From this equation we obtain

$$\widehat{u}(x) = \frac{1}{\beta}(1 - \gamma^2)x - \gamma u \tag{3}$$

If we let $\widehat{u}(u) = u$, then $\beta + \gamma = 1$, i.e., $x * y = \beta x + (1 - \beta)y$ for all $x, y \in K$ and $\widehat{u}(x) = \frac{1}{\beta}[(1 - (1 - \beta)^2)x - \beta(1 - \beta)u] = (2 - \beta)x + (\beta - 1)u$ for all $x \in K$. □

Example 5.9. Let $\beta = \gamma = \frac{1}{2}$ in Proposition 5.8. Then $x * y = \frac{1}{2}(x + y)$ for all $x, y \in K$ and $\widehat{u}(x) = \frac{2}{3}x - \frac{1}{2}u$ is the pseudo inverse function of u for all $x \in K$. Hence $(K, *)$ is completely LR-special.

Remark 5.10. In Theorem 5.7, assume $(K, *)$ has an RL-special element $u \in K$. Then there exists a map $\widehat{u} : K \to K$ such that $u * (\widehat{u}(x) * x) = x$ for all $x \in K$. It

follows that $x = u * (\widehat{u}(x) * x) = \alpha + \beta u + \gamma(\widehat{u}(x) + x) = \alpha + \beta u + \gamma(\alpha + \beta\widehat{u}(x) + \gamma x) = \alpha + \beta u + \alpha\gamma + \beta\gamma\widehat{u}(x) + \gamma^2 x$ and

$$\widehat{u}(x) = \frac{1}{\beta\gamma}[(1-\gamma^2)x - \alpha(1+\gamma) - \beta u] \tag{4}$$

Assume m be a fixed element of $\widehat{u}$, i.e., $\widehat{u}(m) = m$. Then, by (4), we obtain $m = \widehat{u}(m) = \frac{1}{\beta\gamma}[(1-\gamma^2)m - \alpha(1+\gamma) - \beta u]$. It follows that

$$m = \frac{\alpha(1+\gamma) + \beta u}{1 - \gamma^2 - \beta\gamma} \tag{5}$$

when $1 - \gamma^2 - \beta\gamma \neq 0$, i.e., $1 \neq \gamma(\beta + \gamma)$. If $\alpha = 0, \beta = \gamma = \frac{1}{2}$ in (5), then $m = u$, i.e., $\widehat{u}(u) = u$ and $x * y = \frac{1}{2}(x + y)$. If $\alpha = 0, \beta = \gamma = 1$ in (5), then $m = -u$, i.e., $\widehat{u}(u) = u$ and $x * y = x + y$.

Conclusion.

Although the idea of what an inverse of an element means in restricted circumstances, where these are often unique, or unique "on the left" or "on the right", such as in the theory of groups, or the multiplicative semigroups of rings, for example, there has not been a detailed study of "inverse types" that may exist for arbitrary binary systems. Again, it is true that the idea of "inverses" has been touched upon in more general circumstances ([5]) even in the ancestral age of such studies, but it has been limited by the abstractions caused by excess universality, which has not promoted the progress one desires in a more detailed development such as attempted here. In particular, we have dealt with a number of classes of binary systems beyond the standard ones, which generate subsemigroups of $(Bin(X), \Box)$ or which are otherwise of significance, e.g., groups. The methods developed in this paper can easily be applied to other classes of binary systems, and if necessary to other systems of "higher" universal algebra classified types as well.

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