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DETERMINATION OF PARTICULAR SOLUTIONS OF NONHOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS BY DISCRETE DECONVOLUTION

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Se prezintă o nouă metodă de determinare a soluţiilor particulare ale ecuaţiilor diferenţiale liniare neomogene cu coeficienţi constanţi şi parte dreaptă uzuală. Noutatea constă în utilizarea deconvoluţiei discrete la calculul coeficienţilor polinoamelor care apar în formulele soluţiilor. Metoda poate fi uşor implementată pe calculator.

A new method to determine the particular solutions for nonhomogeneous linear differential equations with constant coefficients and usual right parts is presented. The novelty consists in the use of the discrete deconvolution for the computation of the coefficients of the polynomials that appear in the solutions formulae. The method can easy be implemented on computer.

Keywords: nonhomogeneous linear differential equations with constant coefficients, particular solutions, discrete convolution and deconvolution.
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1. Introduction

In an earlier paper [4], the convolution between two sequences was used to determinate the constants that appear in the solution of the Cauchy problem for a homogeneous linear differential equations with constant coefficients and, particularly, the elementary solution of the equation was deduced. In the case of a nonhomogeneous equation of this type, a particular solution was obtained by the integral of convolution between the right part of the equation and the elementary solution of the associated homogeneous equation.

Using the deconvolution of the sequences of the same finite length (see [3]), in the present paper we give a direct method to obtain particular solutions for nonhomogeneous linear differential equations with constant coefficients and standard right parts, without the use of other type of calculi, as would be the deduction by the identification of the coefficients of linear algebraic systems and their solving by recurrence, the determination of the roots of the characteristic equation, or the computation of integrals.

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The deconvolution method presented here results from the usual method of the identification of the coefficients (undeterminate coefficients method). The triangular systems of linear algebraic equations that are obtained for the determination of the constants that appear in the particular solutions of the nonhomogeneous equations are solved by discrete deconvolution. In accord with the type of the right part of the equation and its resonance, a numerical deconvolution formula to obtain a particular solution is given, for every case. The deconvolution method is very suitable to be implemented on computer.

Other applications of the discrete convolution and deconvolution to solution of both initial and bounded value problems for linear difference and differential equations and for computation the roots of polynomials, are given in the papers [1] and [2].

2. Discrete convolution and deconvolution

Let be $a=(a_0;a_1;...;a_n)$ and $b=(b_0;b_1;...;b_n)$ two finite sequences of real or complex numbers, having the same length. We call (see [3]) their *discrete convolution* (*Cauchy product*), the finite sequence

$$c=(c_0;c_1;...;c_n)=a*b \tag{1}$$

having the components given by the formulae

$$c_0=a_0b_0,\ c_1=a_0b_1+a_1b_0\ ,...,\ c_n=\sum\nolimits_{k=0}^{n}a_kb_{n-k}\ . \tag{2}$$

The convolution is associative, commutative, distributive relating to addition and it has $\delta=(1;0;...;0)$ as unit.

The determining of the sequence $b=c/a$ that satisfies the relation (1) , for two given sequences a and c, is called *the discrete deconvolution* of c by a (see also [3]). It can be performed if $a_0\neq 0$, the components of the finite sequence b being given by the relations

$$b_0=c_0/a_0,\ b_1=(c_1-a_1b_0)/a_0,\ \ldots,\ \ b_n=\left(c_n-\sum\nolimits_{k=0}^{n-1}a_{n-k}b_k\right)\Big/a_0 \tag{3}$$

therefore by the algorithm

$$
\begin{array}{llll|llll}
c_0 & c_1 & \cdots & c_n & a_0 & a_1 & \cdots & a_n \\ \cline{5-8}
c_0 & a_1\ b_0 & \cdots & a_n\ b_0 & b_0 & b_1 & \cdots & b_n \\ \cline{1-4}
/ & c_1-a_1\ b_0 & \cdots & c_n-a_n\ b_0 & & & & \\
 & & & c_1-a_1\ b_0\ \ \cdots\ \ a_{n-1}\ b_1 & & & & \\ \cline{1-4}
 & & \cdots & c_n-a_n\ b_0-a_{n-1}\ b_1 & & & &
\end{array}
$$

… …

/ $c_n - \sum_{j=0}^{n-1} a_{n-j}\, b_j$

$c_n - \sum_{j=0}^{n-1} a_{n-j}\, b_j$

/

If $a = (a_0 ; a_1 ; \ldots ; a_n)$ is a finite sequence with $a_0 \neq 0$, the sequence $a^{-1} = \delta / a$ is called *the inverse* of the sequence a with respect to the convolution and we have $c / a = c * a^{-1}$.

3. Linear differential equations with constant coefficients and polynomial right part, without resonance

We consider the problem of finding a particular solution of the linear differential equation of order n,

$$L(x(t)) = P_n(D)(x(t)) = \sum_{k=o}^{n} a_{n-k}\, x^{(k)}(t) = f(t) \tag{4}$$

having a real variable t and complex constant coefficients $a_{n-k}, k = 0,1,\ldots,n$, with $a_o \neq 0$. The main case to which other situations are also reduced, is the one when the right part $f(t)$ of the equation is a polynomial with complex coefficients of an arbitrary degree q,

$$f(t) = Q_q(t) = \sum_{j=0}^{q} b_{q-j}\, t^j \quad . \tag{5}$$

If $a_n \neq 0$, in which case $w = 0$ is not a root of the characteristic equation

$$P_n(w) = \sum_{k=0}^{n} a_{n-k}\, w^k = 0 \ , \tag{6}$$

we say that the equation (4) with the right part (5) has not resonance. In this case we shall determine a particular solution of the equation of the form

$$x(t) = S_q(t) = \sum_{j=0}^{q} c_{q-j}\, t^j , \tag{7}$$

the coefficients of the polynomial being determined by the condition that the function $x(t)$ given by (7) satisfies the equation (4).Then

$$x^{(k)}(t) = S_q^{(k)}(t) = 0,\ \ k > q,$$
$$x^{(k)}(t) = S_q^{(k)}(t) = \sum_{j=k}^{q} c_{q-j} \frac{j!}{(j-k)!} t^{j-k} = \sum_{i=0}^{q-k} c_{q-k-i} \frac{(i+k)!}{i!} t^i \ ,$$

for $k = 0,1,\dots,q$, with the change of the index $i = j - k$. Imposing the convention,

$$a_k = 0,\ \ k < 0 \ , \tag{8}$$

if $x(t)$ is the polynomial given by the relation (7), the equation (4) becomes

$$\sum_{k=0}^{n} a_{n-k} x^{(k)}(t) = \sum_{k=0}^{q} a_{n-k} x^{(k)}(t) = \sum_{k=0}^{q} a_{n-k} \sum_{i=0}^{q-k} c_{q-k-i} \frac{(i+k)!}{i!} t^i =$$
$$= \sum_{i=0}^{q} \sum_{k=0}^{q-i} a_{n-k} c_{q-k-i} \frac{(i+k)!}{i!} t^i = \sum_{i=0}^{q} b_{q-i}\, t^i$$

Identifying the coefficients and performing the change of the index of summation $i = q - j$, the following relations result

$$\sum_{k=0}^{j} a_{n-k} c_{j-k} (q-(j-k))! = b_j (q-j)!\,,\ j = 0,1,\dots,q \tag{9}$$

which represent a triangular linear algebraic system having as unknowns the coefficients c_j, $j = 0,1,\dots,q$. This system is compatible, since $a_n \neq 0$.

In the sequel we shall resolve the algebraic system (9) by discrete deconvolution of the numerical sequences of the same finite length and in the following sections, we shall present the changes that have to made to the method in order to apply it both in the case of resonance and for more general right parts $f(t)$.

Returning to the determination of a solution of the form (7) for the equation (4) with the right part (5), in the case $a_n \neq 0$ and if the condition (8) is satisfied, we consider the sequence of the coefficients $(a_{n-j} : j = 0,1,...,q)$ of the equation (4) and $(b_j : j = 0,1,...,q)$, respectively $(c_j : j = 0,1,...,q)$ of the polynomial (5), respectively (7), and the "normalizations" $(\tilde{b}_j = (q-j)!b_j : j = 0,1,...,q)$, respectively $(\tilde{c}_j = (q-j)!\, c_j : j = 0,1,...,q)$. With these notations, the relation (9) takes the form

$$\left(a_{n-j}\right)*\left(\tilde{c}_j\right)=\left(\tilde{b}_j\right).$$

Thus, the coefficients of the solution (7) can be determined by the following deconvolution formula

$$\left(\tilde{c}_j : j=0,1,\ldots,q\right)=\left(\tilde{b}_j : j=0,1,\ldots,q\right)/\left(a_{n-j} : j=0,1,\ldots,q\right) \qquad (10)$$

Example 1. Find a particular solution of the differential equation

$$3x^{(4)}(t)-7x'''(t)+5x''(t)-3x'(t)+2x(t)=f(t)$$

where a) $f(t)=4t^3-4t^2+12t+5$;

b) $f(t)=2t^6-14t^5+120t^4-640t^3+280t^2+600t+200$

Solution. a) We have $n=4, q=3$, the equation having not resonance, hence we determine a particular solution of the form $x(t)=c_0t^3+c_1t^2+c_3t+c_4$.

Also, $\left(a_{n-j} : j=0,1,\ldots,q\right)=\left(a_{4-j} : j=0,1,2,3\right)=$
$=\left(a_4;a_3;a_2;a_1\right)=\left(2;-3;5;-7\right)$ and $\left(b_j : j=0,1,2,3\right)=\left(4;-4;12;5\right)$, hence $\left(\tilde{b}_j : j=0,1,2,3\right)=\left(3!\,4;-2!\,4;1!\,12;0!\,5\right)=\left(24;-8;12;5\right)$. The sequence $\tilde{c}$ can be obtained by the following deconvolution algorithm:

```
24   -8  12    5 | 2 -3  5 -7
24  -36  60  -84 |------------
-----------------| 12 14 -3  5
 /   28 -48  -84 |
     28 -42   70
     ------------
      /  -6   19
         -6    9
         -------
          /   10
              10
             ---
               /
```

One obtains $\tilde{c}=\left(12;14;-3;5\right)$. Hence $c=\left(\frac{12}{3!};\frac{14}{2!};\frac{-3}{1!};\frac{5}{0!}\right)=$
$=\left(2;7;-3;5\right)$ therefore a particular solution of the equation is

$$x(t)=2t^3+7t^2-3t+5.$$

b) We determine a particular solution of the form
$x(t)=c_0t^6+c_1t^5+c_2t^4+c_3t^3+c_4t^2+c_5t+c_6$. We have
$n=4, q=6, \left(a_{n-j} : j=0,1,...,q\right)=\left(a_{4-j} : j=0,\ldots,6\right)=\left(2;-3;5;-7;3;0;0\right)$,
according to the convention (8),

$b=\left(2;-14;120;-640;280;600;200\right)$,

$\tilde{b}=\left(6!\,2;-5!\,14;4!\,120;-3!\,640;2!\,280;600;200\right)=$

$= (1440; -1680; 2880; -3840; 560; 600; 200)$

By deconvolution, we obtain $\tilde{c} = \tilde{b} / (a_{n-j}) = (720; 240; 0; 0; 40; 0; 0)$, hence

$c = \left(\frac{1}{6!}720; \frac{1}{5!}240; 0; 0; \frac{1}{2!}40; 0; 0\right) = (1; 2; 0; 0; 20; 0; 0)$. Consequently, a particular solution is $x(t) = t^6 + 2t^5 + 20t^2$.

Remark. A variant of the above presented method, useful especially when we have to solve the same equation for several right parts, consists in the determination by deconvolution of the inverse of the sequence $(a_{n-j} : j = 0,1,\dots,q)$ and in finding the sequence $(\tilde{c}_j)$ by the convolution between this inverse and the sequence $(\tilde{b}_j)$. For example, in the case of the above considered equation, we have

$$(a_{n-j})^{-1} = \delta / (a_{n-j}) = (1/2; 3/4; -1/8; -5/16; 55/32; 115/6; -321/128)$$

Performing the convolution between this inverse and the sequences $(\tilde{b}_j)$ from points a) and b), we obtain again the above calculated particular solutions of the equation.

4. Equations with polynomial right parts and resonance

If $a_n = a_{n-1} = \dots = a_{n-m+1} = 0, a_{n-m} \neq 0$, then $w = 0$ is a root of multiplicity m of the characteristic equation (6) and we say that the differential equation (4) has a resonance of order m. In this case the equation (4) has the form

$$\sum_{k=m}^{n} a_{n-k} x^{(k)}(t) = f(t) \tag{11}$$

If we consider the new unknown function

$$y(t) = x^{(m)}(t), \tag{12}$$

the equation takes the form

$$\sum_{k=0}^{n-m} a_{n-m-k} y^{(k)}(t) = f(t). \tag{13}$$

If $f(t)$ is the polynomial given by (5), in accordance with those above mentioned, the equation (13) has the particular solution

$$y(t)=\sum_{j=0}^{q} c_{q-j} t^{j}, \tag{14}$$

the coefficients being determined with the help of the relation

$$\left(\tilde{c}_j\right)=\left(\tilde{b}_j\right)/\left(a_{n-m-j} : j=0,1,\ldots,q\right). \tag{15}$$

Consequently, a particular solution of equation (11) will be the function

$$x(t)=\underbrace{\int\int\ldots\int}_{m} y(t)dt=\underbrace{\int\int\ldots\int}_{m} \sum_{j=0}^{q} c_{q-j} t^{j} dt=$$

$$=\sum_{j=0}^{q} c_{q-j} \frac{j!}{(m+j)!} t^{m+j}=t^{m} S_q(t),$$

where

$$S_q(t)=\sum_{j=0}^{q} C_{q-j} t^{j}, \tag{17}$$

the coefficients of the polynomial being given by the formula

$$C_{q-j}=c_{q-j}\frac{j!}{(m+j)!}=\frac{1}{(m+j)!}\tilde{c}_{q-j},\ j=0,1,\ldots,q. \tag{18}$$

Example 2. Find a particular solution of the differential equation $x^{(4)}(t)+2x^{(3)}(t)=f(t)$ with the right parts a) $f(t)=4t-1$; b) $f(t)=7t^4+6t^3-10t^2+2t$.

Solution. a) We have $n=4, m=3$ (triple resonance), $q=1$ and we determine a particular solution of the form $x(t)=t^3(c_0 t+c_1)$. We have $\left(a_{n-m-j} : j=0,1,\ldots,q\right)=\left(a_{1-j} : j=0,1\right)=\left(a_1; a_0\right)=(2;1), \tilde{b}=b=(4;-1)$.

Using the deconvolution algorithm

4	-1	|	2	1
4	2	|	2	-3/2
/	-3			
	-3			
	/			

one obtains $\tilde{c} = (\tilde{c}_0 ; \tilde{c}_1) = \left(2; -\frac{3}{2}\right)$. From (18), $C_{1-j} = \frac{1}{(3+j)!}\tilde{c}_{1-j}, j = 0,1,$ hence $C_1 = \frac{1}{3!}\tilde{c}_1 = -\frac{1}{3!}\frac{3}{2} = -\frac{1}{4}, C_0 = \frac{1}{4!}\tilde{c}_0 = \frac{1}{4!}2 = \frac{1}{12}$ and one obtains the particular solution $x(t) = t^m S_q(t) = t^m \sum_{j=0}^{q} C_{q-j} t^j = t^3 \sum_{j=0}^{1} C_{1-j} t^j = t^3\left(\frac{t}{12} - \frac{1}{4}\right) = \frac{t^3(t-3)}{12}$.

b) We have $n = 3, m = 3, q = 4$ and we determine a particular solution of the form $x(t) = t^3\left(c_0 t^4 + c_1 t^3 + c_2 t^2 + c_3 t + c_4\right)$. We have $\left(a_{n-m-j} : j = 0,1,\dots,q\right) =$
$= \left(a_{1-j} : j = 0,1,2,3,4\right) = (2;1;0;0;0)$, in conformity with the convention (8),
$b = (7;6;-10;2;0), \tilde{b} = (4!\,7;3!\,6;-2!\,10;2;0) = (168;36;-20;2;0)$.
From the deconvolution algorithm,

168	36	-20	2	0	|	2	1	0	0	0
168	84	0	0	0	|	84	-24	2	0	0
/	-48	-20	2	0						
	-48	-24	0	0						
	/	4	2	0						
		4	2	0						
		/	0	0						

one obtains $\tilde{c} = (84;-24;2;0;0)$ From (18), we have $C_{4-j} = \frac{1}{(3+j)!}\tilde{c}_{4-j}$, $j = 0,\dots,4,$ hence $C_0 = \frac{1}{7!}\tilde{c}_0 = \frac{84}{7!} = \frac{1}{60}$,

$C_1 = \frac{1}{6!}\tilde{c}_1 = -\frac{24}{6!} = -\frac{1}{30}$, $C_2 = \frac{1}{5!}\tilde{c}_2 = \frac{2}{5!} = \frac{1}{60}$, $C_3 = C_4 = 0$, and the particular solution $x(t) = \frac{t^3}{60}\left(t^4 - 2t^3 + t^2\right)$ is obtained .

5. Equations with exp-polynomial right part

The case in which the right part of the equation (4) has the form

$$f(t) = e^{zt} Q_q(t) \tag{19}$$

with z a complex number and $Q_q(t)$ the polynomial given by (5), is reduced to the previous case by change of the unknown function

$$x(t) = e^{zt} y(t) \quad . \tag{20}$$

Using the Leibniz rule for the differentiation of a product of functions, the equation (4) becomes

$$L(x(t)) = \sum_{k=0}^{n} a_{n-k} x^{(k)}(t) = \sum_{k=0}^{n} a_{n-k} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} z^{k-j} e^{zt} y^{(j)}(t) = e^{zt} Q_q(t). \tag{21}$$

Because $\frac{d^j}{dz^j} z^k = 0,\ j > k$ and $\frac{d^j}{dz^j} z^k = \frac{k!}{(k-j)!} z^{k-j},\ j = 0,1,\ldots,k,$ the equation (21) turns as follows

$$\tilde{L}(y(t)) = \sum_{k=0}^{n} a_{n-k} \sum_{j=0}^{n} \frac{1}{j!} \frac{d^j}{dz^j}\left(z^k\right) y^{(j)}(t) =$$

$$= \sum_{j=0}^{n} \frac{1}{j!} P_n^{(j)}(z) y^{(j)}(t) = \sum_{j=0}^{n} A_{n-j} y^{(j)}(t) = Q_q(t). \tag{22}$$

The equation (22) is of the type considered in the parts 2 and 3 and it has the characteristic equation of the form

$$\tilde{P}_n(w) = \sum_{j=0}^{n} A_{n-j} w^j = 0 \tag{23}$$

with

$$A_{n-j} = \frac{1}{j!} P_n^{(j)}(z),\ j = 0,1,\ldots,n \quad . \tag{24}$$

The number z is a root of multiplicity m of the characteristic equation (6) if $P_n(z)=P_n'(z)=\ldots=P_n^{(m)}(z)=0$ and $P_n^{(m+1)}(z)\neq 0$. In this case the number $w=0$ is a root of multiplicity m of the characteristic equation (23). We consider also $m=0$ if $P_n(z)\neq 0$, hence if $w=0$ is not a root of the characteristic equation (23). In accordance with those above presented at parts 2 and 3, a solution of the equation (22) is given by the formula $y(t)=t^m S_q(t)$, hence the differential equation (4) with the right part (19) has the particular solution

$$x(t)=t^m e^{zt} S_q(t), \tag{25}$$

the coefficients of the polynomial $S_q(t)$ being determined by the relation (10) if $m=0$, respectively by (15) and (18) if $m\neq 0$, with the observation that in the present case the sequence $(a_{n-m-j} : j=0,1,...,q)$ of the coefficients of equation (4) must be replaced by the sequence $(A_{n-m-j} : j=0,1,...,q)$ of the coefficients of the equation (22).

Example 3. Find a particular solution of the differential equation

$$L(x(t))=x'''(t)-x''(t)-4x'(t)+4x(t)=2t^2-4t-1+\left(2t^2+5t+1\right)e^{2t}$$

Solution. We consider the equation $L(x(t))=f_j(t), j=1,2$, with $f_1(t)=2t^2-4t-1$ respectively $f_2(t)=\left(2t^2+5t+1\right)e^{2t}$. In the case of the first equation, $n=3, q=2, m=0$, because the number $w=0$ is not a root of the characteristic equation $P_3(w)=w^3-w^2-4w+4$, hence the first differential equation is not in resonance and we determine its particular solution of the form $x_1(t)=c_0t^2+c_1t+c_2$

We have

$(a_{3-j} : j=0,1,2)=(4;-4;-1)\, b=(2;-4;-1)\,, \tilde{b}=(2!\ 2;-4;-1)=(4;-4;-1)$.

From the deconvolution algorithm,

$$\begin{array}{rrr|rrr} 4 & -4 & -1 & 4 & -4 & -1 \\ \cline{4-6} 4 & -4 & -1 & 1 & 0 & 0 \\ \cline{1-3} 0 & 0 & 0 & & & \end{array}$$

it results that $\tilde{c}=(1;0;0), c=\left(\frac{1}{2!};0;0\right)=\left(\frac{1}{2};0;0\right)$, hence a solution for the first equation is $x_1(t)=\frac{1}{2}\cdot t^2$. For the second equation we have $n=3, q=2, m=1$,

because the number $w = z = 2$ is a simple root of the characteristic equation $P_3(w) = 0$, the differential equation being now with simple resonance and we determine its particular solution of the form $x_2(t) = t\left(c_0 t^2 + c_1 t + c_0\right)e^{2t}$. In accordance to (24), we have
$\left(A_{n-m-j} : j = 0,1,\ldots,q\right) = \left(A_{2-j} : j = 0,1,2\right) = \left(A_2 ; A_1 ; A_0\right) = (4;5;1)$

From the deconvolution algorithm

```
 4 5 1 | 4 5 1
 4 5 1 | 1 0 0
 / 0 0
```

it results that $\tilde{c} = \left(\tilde{c}_0 ; \tilde{c}_1 ; \tilde{c}_2\right) = (1;0;0)$. In accordance to (18), we have
$C_{2-j} = \dfrac{1}{(1+j)!}\tilde{c}_{2-j}, j = 0,1,2,$ hence $C_0 = \dfrac{1}{3!}\cdot\tilde{c}_0 = \dfrac{1}{6}, C_1 = \dfrac{1}{2!}\cdot\tilde{c}_1 = 0, C_2 = \tilde{c}_2 = 0$.

Therefore a solution for the second equation is $x_2(t) = \dfrac{1}{6}t^3 e^{2t}$, a particular solution for the second differential equation being

$$x(t) = x_1(t) + x_2(t) = \frac{1}{2}t^2 + \frac{1}{6}t^3 e^{2t}.$$

6. Equations with trig-exp-polynomial right parts

If the differential equation (4) has real coefficients and its right part is of the form

$$f(t) = e^{\alpha t}\left[Q_q(t)\cos(\beta t) + R_r(t)\sin(\beta t)\right] \qquad (26)$$

were α, β are real numbers and $Q_q(t)$, $R_r(t)$ polynomials of degree q respectively r, having real coefficients, then the equation can be reduced to the above case and a particular solution can be determined with the help of the discrete deconvolution.

Firstly we observe that, by completion with null terms, the polynomials Q and P can be considered as having the same degree $d = \max(q, r)$.

We will find some particular solutions $x_{1,C}(t), x_{2,C}(t), x_{1,S}(t), x_{2,S}(t)$ of the equation (4) for any right part

$$f_{1,C}(t)=e^{\alpha t}Q_d(t)\cos(\beta t),\ f_{2,C}(t)=e^{\alpha t}R_d(t)\cos(\beta t),$$
$$f_{1,S}(t)=e^{\alpha t}Q_d(t)\sin(\beta t),\ f_{2,S}(t)=e^{\alpha t}R_d(t)\sin(\beta t). \tag{27}$$

In this case, the functions
$x_1(t)=x_{1,C}(t)+i\,x_{1,S}(t),\ x_2(t)=x_{2,C}(t)+i\,x_{2,S}(t)$
are particular solutions of the equation (4) for the right parts

$$f_1(t)=f_{1,C}(t)+i\,f_{1,S}(t)=e^{zt}Q_d(t)\ ,\quad f_2(t)=f_{2,C}(t)+i\,f_{2,S}(t)=e^{zt}R_d(t)\ , \tag{28}$$

respectively, where $z=\alpha+i\beta$. The differential equation (4) with the right hand parts of the form (28) is of the type considered in the section 4. In accordance with those presented there, it results that the differential equation (4) with the right parts given by the relation (28) has a particular solution of the form

$$x_1(t)=t^m e^{zt}S_d(t)\quad ,\qquad x_2(t)=t^m e^{zt}T_d(t), \tag{29}$$

respectively, where m is the multiplicity of the number z as a root of the characteristic equation (6), particularly $m=0$ if z is not a root of this equation. Here $S_d(t),T_d(t)$ are polynomials of degree d , their coefficients being determined by deconvolution with the numerical sequence $\left(A_{n-m-j}:j=0,1,...,q\right)$ as it was presented in the above section.

Since $f(t)=f_{1,C}(t)+f_{2,S}(t)=\operatorname{Re}f_1(t)+\operatorname{Im}f_2(t)$, a particular solution of the equation (4) with the right part (26) is the function

$$x(t)=x_{1,C}(t)+x_{2,S}(t)=\operatorname{Re}x_1(t)+\operatorname{Im}x_2(t)=\operatorname{Re}\left(t^m e^{zt}S_d(t)\right)+\operatorname{Im}\left(t^m e^{zt}T_d(t)\right)=$$
$$=t^m e^{\alpha t}\{\operatorname{Re}[(\cos(\beta t)+i\sin(\beta t))(\operatorname{Re}S_d(t)+i\operatorname{Im}S_d(t))]+$$
$$+\operatorname{Im}[(\cos(\beta t)+i\sin(\beta t))(\operatorname{Re}T_d(t)+i\operatorname{Im}T_d(t))]\}=$$
$$=t^m e^{\alpha t}\left[(\operatorname{Re}S_d(t)+\operatorname{Im}T_d(t))\cos(\beta t)+(\operatorname{Re}T_d(t)-\operatorname{Im}S_d(t))\sin(\beta t)\right]\ . \tag{30}$$

Example 4. Find a particular solution of the differential equation
$Lx(t)=x''(t)-4x'(t)+13x(t)=e^t\left[\left(-2t^2+7t-1\right)\cos(2t)+\left(3t^2-1\right)\sin(2t)\right]$

Solution. We consider the equations $Lx(t)=f_1(t)=e^{zt}\left(-2t^2+7t-1\right)$ and

$Lx(t) = f_2(t) = e^{zt}(3t^2 - 1)$, for $z = 1 + 2i$, that is not a root of the characteristic equation $P_2(w) = w^2 - 4w + 13 = 0$, hence the two considered equations have not resonance. By the change of the unknown function $x(t) = e^{zt}y(t)$ and by the notation $\tilde{L}y(t) = A_0 y''(t) + A_1 y'(t) + A_2 y(t) = y''(t) + (-2 + 4i)y'(t) + (6 - 4i)y(t)$, since $A_0 = \frac{1}{2!}P_2''(z) = 1,\ A_1 = \frac{1}{1!}P_2'(z) = 2z - 4 = -2 + 4i$,

$A_2 = \frac{1}{0!}P_2(z) = z^2 - 4z + 13 = 6 - 4i$, we obtain the differential equations $\tilde{L}(y(t)) = Q_2(t) = -2t^2 + 7t - 1$ and $\tilde{L}(y(t)) = R_2(t) = 3t^2 - 1$.

The finite sequences of the coefficient of polynomials from the right part of these equations being $b' = (-2; 7; 1)$,

$\tilde{b}' = (-4; 7; -1), b'' = (3; 0; -1), \tilde{b}'' = (6; 0; -1)$, and the finite sequence of the coefficients of the left part of the equations being

$$(A_{2-j} : j = 0,1,2) = (A_2; A_1; A_0) = (6 - 4i; -2 + 4i; 1),$$

one obtains by deconvolution

$$\frac{\tilde{b}'}{(A_{2-j})} = (\tilde{c}_0'; \tilde{c}_1'; \tilde{c}_2') = \left(-\frac{2(3 + 2i)}{13}; \frac{157 + 174i}{169}; \frac{709 + 282i}{2197}\right)$$

$$\frac{\tilde{b}''}{(A_{2-j})} = (\tilde{c}_0''; \tilde{c}_1''; \tilde{c}_2'') = \left(\frac{3(3 + 2i)}{13}; \frac{3(29 + 2i)}{169}; \frac{282 - 709i}{2197}\right),$$

hence the coefficients

$$c_0' = \frac{1}{2}\tilde{c}_0' = -\frac{3 + 2i}{13},\ c_1' = \tilde{c}_1' = \frac{157 + 174i}{169},\ c_2' = \tilde{c}_2' = \frac{709 + 282i}{2197},$$

$$c_0'' = \frac{1}{2}\tilde{c}_0'' = \frac{9 + 6i}{26},\ c_1'' = \tilde{c}_1'' = \frac{87 + 6i}{169},\ c_2'' = \tilde{c}_2'' = \frac{282 - 709i}{2197}.$$

are obtained. It results that the two equations have respectively the particular solutions

$$x_1(t) = e^{zt}S_2(t), x_2(t) = e^{zt}T_2(t),\ \text{where}\ e^{zt} = e^t[\cos(2t) + i\sin(2t)],$$

$$S_2(t) = -\frac{3 + 2i}{13}t^2 + \frac{157 + 174i}{338}t + \frac{709 + 282i}{2197},$$

$$T_2(t) = \frac{9 + 6i}{26}t^2 + \frac{87 + 6i}{169}t + \frac{282 - 709i}{2197}.$$ Therefore

$$\operatorname{Re} x_1(t) = e^t\left[\left(-\frac{3}{13}t^2 + \frac{157}{338}t + \frac{709}{2197}\right)\cos(2t) + \left(\frac{2}{13}t^2 - \frac{87}{169}t - \frac{282}{2197}\right)\sin(2t)\right],$$

$$\operatorname{Im} x_2(t) = e^t\left[\left(\frac{3}{13}t^2 + \frac{6}{169}t - \frac{709}{2197}\right)\cos(2t) + \left(\frac{9}{26}t^2 + \frac{87}{169}t + \frac{282}{2197}\right)\sin(2t)\right],$$

hence the initial differential equation has the particular solution

$$x(t) = \operatorname{Re} x_1(t) + \operatorname{Im} x_2(t) = \frac{1}{2}\, e^t \left[t\cos(2t) + t^2\sin(2t)\right].$$

R E F E R E N C E S

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