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CONVERGENCE TO A COMPACT SET IN FUNCTIONAL SPACES

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In this paper we introduce a convergence concept for compact subset of a sequence of functions. This convergence is called Γ-statistical uniform convergence. We also establish necessary and sufficient conditions for Γ-statistical uniform convergence to a compact set of cluster functions of a sequence of functions in functional spaces. Furthermore, we investigate some properties and applications of new type convergence.

Keywords: Set of statistically cluster points, a sequence of functions, Γ-statistical uniform convergence.

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1. Introduction

In [1], Arzelá investigated a necessary and sufficient condition under which the pointwise limit of a sequence of real valued continuous functions on a compact interval is continuous. Several generalizations and applications of this notion have been investigated [2], [13], [14], [15], [16], [17]. In [3] Beer and Levi proposed a new approach to this investigation, on a metric space (X, d), through the notion of strong uniform convergence on bornologies, when this bornology reduces to that of all nonempty finite subsets of X. In [5], a direct proof of the equivalence of Arzelá, Alexandroff, and Beer-Levi conditions was given. If a sequence of measurable functions defined on an interval with Lebesgue measure is pointwise statistically convergent to f, then it is statistically convergent in measure to f, it was proved by Steinhaus [31] and Fast in [10]. In [12] Fridy introduced the concepts of statistical (or shortly, st) limit points and st- cluster points of a number sequence and given some properties of the sets of st-limit points and st-cluster points of a sequence of real numbers. We introduced in [28] the concept of Γ st - convergence using the set of cluster points in a $\mathbb{R}^m$ space. Later, in [20], [22], [26], [30], we introduced some applications of the concept of Γ st - convergence, depending on the concepts of a compact set of cluster points of a sequence in $\mathbb{R}^m$ space, in Turnpike theory. We studied an asymptotic behaviour of optimal paths and optimal controls in problem of optimal control in discrete time [21]. St- convergence and Γ st - convergence are very useful and more general tools than their ordinary counterparts, that is, when we cannot model the behaviour of a sequence via the tools of ordinary convergence, we can benefit from st - convergence and Γ st- convergence [32]. There are many papers on st - convergence in the literature, and now it seems nearly impossible to list all of them. Nevertheless, for the definition and various properties of st - convergence in a general Hausdorff topological space, please see [19].

We continue to develop our work in [27] and [28]. For this purpose, first, we compare some of the results related to the concept of st - cluster point introduced by Fridy [12] and developed by Pehlivan et al. [28], with our results obtained in the more complicated functions spaces. Next, using the theorem of The Arzela-Ascoli's, we introduce the concept of the compact set of st-uniform cluster functions in $C(X)$, called Γ-st-uniform convergence.

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Let us outline what we have done in this work. In the second section some basic definitions related to the theory are recalled and given some preliminary concepts of the sets of st - limit and cluster points. In Section 3, we study st-uniform limit (*in short, sul*) functions and st-uniform cluster (*in short, suc*) functions for sequences of functions in Y^X. We investigate some of the properties of these sets. In Section 4, we continue this study the set of all suc - functions Γ_F^u is nonempty and compact in $C(X)$ and provide additional information about the compact set of suc functions in $C(X)$. Finally, we propose a notion of a new type convergence for a sequence of functions which is if $F = \{f_k\}_{k\in\mathbb{N}}$ be a sequence $st-$ uniformly bounded of functions in $C(X)$ then $F = \{f_k\}_{k\in\mathbb{N}}$ is Γ-st-uniformly convergent to the set Γ_F^u.

2. **Notations and Definitions**

We list some of the basic definitions and notations related to the theory of st-convergence. Let $A \subset \mathbb{N}$ and

$$d_n(A) = \frac{1}{n}\sum_{k=1}^{n} \chi_A(k).$$

If $\lim_{n\to\infty} d_n(A)$ exists, then it called the natural density of A and denoted by $\delta(A)$ (see, e.g. [8], [9], [25]). A set $A \subset \mathbb{N}$ is said to be statistically dense if $\delta(A) = 1$, and a subsequence $\{x_k\}_{k\in A}$ of a sequence $\{x_k\}_{k\in\mathbb{N}}$ is called statistically dense if $\delta(A) = 1$ (see [19]). The natural density may not exists for a set A, but the upper density of A always exists, and is defined by

$$\overline{\delta}(A) = \limsup_{n\to\infty} d_n(A).$$

The statistical convergence is based on the concept of the natural density of $A \subset \mathbb{N}$. This concept was studied and generalized in various directions by Salat [29], Fridy [11], and the others [18]. Let (X,ρ) be a metric space. A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ in X is said to be statistically convergent to a point η in X, and we write $x_k \overset{st}{\to} \eta$, provided that

$$\delta\left\{k \in \mathbb{N} : \rho(x_k,\eta) \geq \epsilon\right\} = 0.$$

Definition 2.1. *Let (X,ρ) be a metric space. Then a point $\eta \in X$ is said to be a st- cluster point of x provided that for every $\epsilon > 0$ we have $\overline{\delta}\{k \in \mathbb{N} : \rho(x_k,\eta) < \epsilon\} > 0$. The set of all st-cluster points of x is denoted by Γ_x.*

Definition 2.2. *Let (X,ρ) be a metric space. An element $\eta \in X$ is said to be a st- limit point of x provided that there is a set $K = \{k_j : k_1 < k_2 < \ldots\} \subset \mathbb{N}$ such that $\delta(K) > 0$ and $\lim_{j\to\infty} x_{k_j} = \eta$. Denoted by (Λ_x) the set of all st-limit points of x.*

The concepts of the sets of st- cluster and st-limit points a sequence of real numbers $x = \{x_k\}_{k\in\mathbb{N}}$ were defined and studied some properties of these sets by Fridy [12].

Now we consider the concepts related to st- uniform convergence of a sequence of functions on metric spaces. We investigate the sets of sul - functions and suc - functions. We also investigate some of the properties of these sets.

3. **The Sets of st- uniform limit and cluster for sequences of functions**

We can write the definitions of st-uniform convergence in the following [2]: Let (X,ρ_1) and (Y,ρ_2) be metric spaces. Let Y^X be the set of all functions from X to Y and $C(X,Y)$ be the set of all continuous functions from X to Y.

Next let us define st-uniform convergence of $F = \{f_k\}_{k\in\mathbb{N}}$ to f. We write it as $\sup_{x\in X}\rho_2\left(f_k(x), f(x)\right) \overset{st}{\to} 0$. A sequence $F = \{f_k\}_{k\in\mathbb{N}}$ of functions in Y^X is said to be statistically uniformly convergent to a function f in Y^X in symbol: $F \overset{st}{\rightrightarrows} f$ if $(\forall \epsilon > 0)$

$$\delta\{k : \text{for all} \quad x \in X, \quad \rho_2\left(f_k(x), f(x)\right) \geq \epsilon\} = 0. \tag{1}$$

Now we can give the definitions of st-uniform limit and cluster function

Definition 3.1. *A function f in Y^X is said to be a sul - function of a sequence $F = \{f_k\}$ of functions in Y^X provided that there is a set $P = \{p_1 < p_2 < \ldots < p_k < \ldots\} \subset \mathbb{N}$ such that $\delta(P) > 0$ and $f_{p_k}(x) \rightrightarrows f(x)$ for all $x \in X$. That is for every $\epsilon > 0$ we have*

$$\delta\{k : \text{for all} \quad x \in X, \quad \rho_2\left(f_{p_k}(x), f(x)\right) < \epsilon\} > 0. \tag{2}$$

Then the function f is a sul- function and let Λ_F^u denotes the set of sul- functions of a sequence $F = \{f_k\}$ of functions.

Definition 3.2. *A function f in Y^X is said to be a suc - function of a sequence $F = \{f_k\}$ of functions in Y^X if for every $\epsilon > 0$ we have*

$$\delta\{k : \text{for all} \quad x \in X, \quad \rho_2\left(f_k(x), f(x)\right) < \epsilon\} > 0.$$

Let Γ_F^u denotes the set of st-uniform cluster (in short suc) functions of a sequence $F = \{f_k\}$ of functions.

Example 3.1. *Let $F = \{f_k\}$ be the sequence of functions in $\mathbb{R}^{\mathbb{R}}$ defined by*

$$f_k(x) = \begin{cases} \gamma(x), & \text{if } k \text{ is prime} \\ \frac{1}{k(1+x^2)}, & \text{otherwise} \end{cases}$$

for $\forall x \in \mathbb{R}$. Then we have $\delta\{k : \text{for all} \quad x \in \mathbb{R} \quad f_k(x) \geq \epsilon\} = 0$. Therefore the sequence $\{f_k\}$ is st-uniform convergent to the constantly zero function. Since the set of prime numbers has natural density 0, we have $f_k \overset{st}{\rightrightarrows} 0$, but a sequence $\{f_k\}$ of functions is not uniform convergence. $\Gamma_F^u = \{f(x) = 0\} = \{\underline{0}\}$. Let L_F^u be the set of uniform limit functions of a sequence $F = \{f_k\}$ of functions and so $L_F^u = \{\underline{0}, \gamma(x)\}$.

We also prove easily in the following results.

Theorem 3.1. *Let L_F^u be the set of all uniform limit functions of a sequence $F = \{f_k\}$ of functions in Y^X. Then $\Lambda_F^u \subseteq \Gamma_F^u \subseteq L_F^u$.*

Proof. If the function $\tilde{f}$ is a sul- function, then by definition there is a set $P = \{p_1 < p_2 < \ldots < p_k < \ldots\} \subset \mathbb{N}$ such that $\delta(P) > 0$ and $f_{p_k}(x) \rightrightarrows \tilde{f}(x)$ for all $x \in X$. We have

$$\delta\{k : \text{for all} \quad x \in X, \rho_2\left(f_{p_k}(x), \tilde{f}(x)\right) < \epsilon\} > 0.$$

Since $A = \{k : \text{for all}\, x \in X,\ \rho_2\left(f_k(x), \tilde{f}(x)\right) < \epsilon\}$ and $B = \{k : \text{for all}\, x \in X,\ \rho_2\left(f_{p_k}(x), \tilde{f}(x)\right) < \epsilon\}$, we have $A \supset B$. Hence $\delta\{A\} \geq \delta\{B\}$ and so

$$\delta\{k : \text{for all} \quad x \in X,\ \rho_2\left(f_k(x), \tilde{f}(x)\right) < \epsilon\} > 0.$$

This mean that $\tilde{f} \in \Gamma_F^u$ therefore $\Lambda_F^u \subseteq \Gamma_F^u$. The second part of inclusion is clear and the proof is complete. □

Corollary 3.1. *If $F = \{f_k\}_{k\in\mathbb{N}}$ is a sequence of functions in Y^X and $F \overset{st}{\rightrightarrows} f$ for $x \in X$. Then $\Lambda_F^u = \Gamma_F^u = \{f\}$.*

Theorem 3.2. *Let $F = \{f_k\}_{k\in\mathbb{N}}$ and $H = \{h_k\}_{k\in\mathbb{N}}$ are two sequences of functions in Y^X such that $\delta\{k \in \mathbb{N} : \text{for all} \quad x \in X,\ f_k(x) \neq h_k(x)\,\} = 0$ then $\Gamma_F^u = \Gamma_H^u$ and $\Lambda_F^u = \Lambda_H^u$ where the set $\{k \in \mathbb{N} : \text{for all} \quad x \in X,\ f_k(x) \neq h_k(x)\,\}$ (in short $\{k \in \mathbb{N} : f_k \neq h_k\,\}$ or $F \equiv_{st} H$).*

Proof. Let $\tilde{f}$ be a suc - function of the sequence $F = \{f_k\}_{k\in\mathbb{N}}$. Then from the definition we have $\delta\{k \in \mathbb{N} : \text{for all} \quad x \in X, \quad \rho_2\left(f_k(x), \tilde{f}(x)\right) < \epsilon\} > 0$. But note that if $\delta\{k \in \mathbb{N} : \text{for all} \quad x \in X, \quad \rho_2\left(h_k(x), \tilde{f}(x)\right) < \epsilon\} = 0$. Then since $\{k \in \mathbb{N} : \text{for all} \quad x \in X, \quad \rho_2\left(f_k(x), \tilde{f}(x)\right) < \epsilon\} \subset$
$\{k \in \mathbb{N} : \text{for all} \quad x \in X, \ \rho_2\left(h_k(x), \tilde{f}(x)\right) < \epsilon\} \cup \{k \in \mathbb{N} : f_k \neq h_k\}$ so by our assumption $\delta\{k \in \mathbb{N} : \text{for all} \quad x \in X, \quad \rho_2\left(f_k(x), \tilde{f}(x)\right) < \epsilon\} > 0$ which is a contradiction. Therefore $\delta\{k \in \mathbb{N} : \text{for all} \quad x \in X, \quad \rho_2\left(h_k(x), \tilde{f}(x)\right) < \epsilon\} > 0$ and so $\tilde{f} \in \Gamma_H^u$. Similarly we can show that $\Gamma_H^u \subset \Gamma_F^u$ and so the equality is proved.

Let $\tilde{f}$ be a sul - function of the sequence $F = \{f_k\}_{k\in\mathbb{N}}$, so that there exists a set $K = \{k_1 < k_2 < k_3 < \ldots < k_n < \ldots\} \subset \mathbb{N}$ such that $\delta(K) > 0$ and $f_{k_n}(x) \rightrightarrows \tilde{f}(x)$ for all $x \in X$ in uniform topology. Since $\delta\{k \in \mathbb{N} : f_k \neq h_k\} = 0$ and

$$\{k_n \in \mathbb{N} : f_{k_n} \neq h_{k_n}\} \subseteq \{k \in \mathbb{N} : f_k \neq h_k\}$$

then define $M = \{m_k \in \mathbb{N} : f_{m_k} = h_{m_k}\}$ and so $\delta(M) \neq 0$. Denoting by $\{m_k\}_{k\in\mathbb{N}}$ the canonical enumeration of M, we obtain $h_{m_k}(x) \rightrightarrows \tilde{f}(x)$ that is the function $\tilde{f}$ is a sul-function of the sequence $H = \{h_k\}_{k\in\mathbb{N}}$. Hence $\tilde{f} \in \Lambda_H^u$. By the arbitrariness of $\tilde{f} \in \Lambda_F^u$ we have $\Lambda_F^u \subseteq \Lambda_H^u$. Similarly we can prove $\Lambda_H^u \subseteq \Lambda_F^u$. Thus $\Lambda_F^u = \Lambda_H^u$. □

4. Properties of the set of suc- functions in $C(X)$

Now we study some properties of the set of suc - functions in $C(X)$. Let (X, ρ) be a compact metric space and a sequence of functions $F = \{f_k\}_{k\in\mathbb{N}}$ in $C(X, \mathbb{R}) = C(X)$. We get the uniform topology on $C(X)$. The uniform topology on $C(X)$ is defined by

$$B_\varepsilon(g) = \{f \in C(X) : \text{for all } x \in X \quad |g(x) - f(x)| < \varepsilon\}$$

as a base for each $f \in C(X)$ and $\varepsilon > 0$.

Lemma 4.1. *Let Γ_F^u be a set of all suc- functions. Then Γ_F^u is a closed set in $C(X)$.*

Proof. Let $\tilde{f} \in \overline{\Gamma_F^u}$. Then $B_\varepsilon(\tilde{f}) \cap \Gamma_F^u \neq \emptyset$. Therefore there exists a function $f \in B_{\frac{\varepsilon}{2}}(\tilde{f}) \cap \Gamma_F^u$. Since $f \in B_{\frac{\varepsilon}{2}}(\tilde{f})$ for every $x \in X$

$$|\tilde{f}(x) - f(x)| < \frac{\varepsilon}{2}$$

holds. Since $f \in \Gamma_F^u$ we have

$$\mathcal{B} = \{k \in \mathbb{N} : \text{ for all } x \in X, \quad |f_k(x) - f(x)| < \frac{\varepsilon}{2}\}$$

and $\delta(\mathcal{B}) > 0$. Then for each $k \in \mathcal{B}$ and all $x \in X$ we have

$$|f_k(x) - \tilde{f}(x)| \leq |f_k(x) - f(x)| + |f(x) - \tilde{f}(x)| < \varepsilon.$$

Thus $\tilde{f} \in \Gamma_F^u$. Conclude that Γ_F^u is a closed set in $C(X)$. □

Now the question arises the Λ_F^u closed set in $C(X)$. The following example shows that the set Λ_F^u is not closed set.

Example 4.1. *Let us define the functions h_j from $\mathbb{R}$ to $\mathbb{R}$ for all $j \in \mathbb{N}$ by*

$$h_j(x) = \begin{cases} \frac{1}{j} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}.$$

Let $V_j = \{2^{j-1}(2q+1) : q \in \mathbb{N}\}$ *and consider the sequence of functions* $F = \{f_k\}_{k\in\mathbb{N}}$ *be defined by* $f_k(x) = h_j(x)$ *if* $k \in V_j$*. For each* $j \in \mathbb{N}$ *we have* $\delta(V_j) = \frac{1}{2^j} > 0$*. Then for each* $j \in \mathbb{N}$ *the subsequences* $\{f_k\}_{k\in V_j}$ *are uniformly convergent to* h_j *on* $\mathbb{R}$*. Hence we have* $h_j \in \Lambda_F^u$ *for each* $j \in \mathbb{N}$*. It is clear that* $f \notin \Lambda_F^u$ *where* $f(x) = 0$ *for every* $x \in \mathbb{R}$*. Since for each* $\varepsilon > 0$ *there exists a* $h_j \in B_\varepsilon(f) \cap \Lambda_F^u$*,* f *is a cluster function of the set* Λ_F^u*. However, since* $f \notin \Lambda_F^u$ *the set* Λ_F^u *is not closed.*

Now we need the The Arzelá Ascoli Theorem .

Theorem 4.1. *(Arzelá -Ascoli's Theorem.) Let* (X, ρ) *be a compact metric space, and let* $\mathcal{K}$ *be a subset of* $C(X)$*. Then* $\mathcal{K}$ *is compact if and only if* $\mathcal{K}$ *is closed, uniformly bounded, and equicontinuous (see* [6], [7], [24] *).*

Lemma 4.2. *Let* $F = \{f_k\}_{k\in\mathbb{N}}$ *be a sequence of functions defined on a compact metric space* X *such that* $f_k \in C(X)$ *for* $k \in \mathbb{N}$ *and* $\mathcal{A} \subset C(X)$ *be a compact set. If* $\delta\{k \in \mathbb{N} : f_k \in \mathcal{A}\} > 0$ *then* $\mathcal{A} \cap \Gamma_F^u \neq \emptyset$ *.*

Proof. Suppose first that $\mathcal{A}$ is compact subset of $C(X)$. Let $\mathcal{A} \cap \Gamma_F^u = \emptyset$, every function $f \in \mathcal{A}$ is not a suc-function. There is a positive number $\varepsilon = \varepsilon(f) > 0$ such that

$$\delta\{k \in \mathbb{N} : \text{for all}\, x \in X, |f_k(x) - f(x)| < \varepsilon\} = \delta\{k \in \mathbb{N} : \sup_{x\in X} |f_k(x) - f(x)| < \varepsilon\} = 0$$

Let $B_\varepsilon(f) = \{h \in C(X) : \text{for all } x \in X, \quad |f(x) - h(x)| < \varepsilon\}$ be an open set. Since $\mathcal{A}$ is a compact set and so there exists a finite subcover of $\mathcal{A}$. There exists $h_1, h_2, \ldots, h_n$ in $\mathcal{A}$ such that $\mathcal{A} \subset \bigcup_{i\leq n} B_\varepsilon(h_i)$ where

$$B_\varepsilon(h_i) = \{g \in C(X) : \text{for all} \quad x \in X, \quad |h_i(x) - g(x)| < \varepsilon\}$$

for $i = 1, 2, \ldots, n$.
A finite set of continuous functions is obviously equicontinuous, and therefore there exists a η for which this implication is valid. That is $\rho(u, v) < \eta$ then $|h_i(u) - h_i(v)| < \varepsilon$ for $1 \leq i \leq n$,

$$\begin{aligned}\{k \in \mathbb{N} : f_k \in \mathcal{A}\} &\subseteq \bigcup_{i\leq n}\{k \in \mathbb{N} : f_k \in B_\varepsilon(h_i)\} \\ &= \bigcup_{i\leq n}\{k \in \mathbb{N} : \text{for all} \quad x \in X, \quad |f_k(x) - h_i(x)| < \varepsilon\}.\end{aligned}$$

And so

$$\delta\{k \in \mathbb{N} : f_k \in \mathcal{A}\} \leq \sum_{i=1}^{n} \delta\{k \in \mathbb{N} : \text{for all} \quad x \in X, \quad |f_k(x) - h_i(x)| < \varepsilon\}.$$

Since $\delta\{k \in \mathbb{N} : \text{for all} \quad x \in X, |f_k(x) - h_i(x)| < \varepsilon\} = 0$ for every $h_i \in \mathcal{A}$, it follows $\delta\{k \in \mathbb{N} : f_k \in \mathcal{A}\} = 0$, which contradics to $\delta\{k \in \mathbb{N} : f_k \in \mathcal{A}\} > 0$ then $\mathcal{A} \cap \Gamma_F^u \neq \emptyset$ □

Remark 4.1. If the set $\mathcal{A}$ is not uniformly bounded, closed and equicontinuous this theorem does not hold.

Example 4.2. *We define the sequence* $F = \{f_k\}_{k\in\mathbb{N}}$ *by* $f_k(x) = 1/k$ *for every* $x \in [0, 1]$ *and each* $k \in \mathbb{N}$*. Let* $\mathcal{A} = \{f_k : k \in \mathbb{N}\}$*. The set* $\mathcal{A}$ *is bounded, equicontinuous, but not closed and so it is not compact. We have*

$$\{k \in \mathbb{N} : f_k \in \mathcal{A}\} = \mathbb{N},$$

but $\Gamma_F^u = \{f\}$ *where* $f(x) = 0$ *for every* $x \in [0, 1]$ *and so* $\mathcal{A} \cap \Gamma_F^u = \emptyset$*.*

Definition 4.1. *Let $F = \{f_k\}_{k\in\mathbb{N}}$ be a sequence of functions defined on X such that $f_k \in C(X)$ for $k \in \mathbb{N}$. If the sequence $F = \{f_k\}_{k\in\mathbb{N}}$ has a subsequence $\{f_{p_k}\}$ such that the set $\{f_{p_k} : k \in \mathbb{N}\}$ is contained in a compact set $\mathcal{P}$ in $C(X)$ and $\delta\{k : f_{p_k} \in \mathcal{P}\} > 0$ then we say that $F = \{f_k\}_{k\in\mathbb{N}}$ is a sequence st $-$uniformly bounded of functions defined in $C(X)$. Note that in this case we have $\delta\{k : f_k \notin \mathcal{P}\} = 0$ or $\delta\{k \in \mathbb{N} : f_k \in \mathcal{P}\} = 1$.*

Thus we have the following

Theorem 4.2. *If $F = \{f_k\}_{k\in\mathbb{N}}$ is a st - uniformly bounded sequence of functions in $C(X)$ then the set Γ_F^u is a nonempty and compact in $C(X)$.*

Proof. Let $\{f_{p_k}\}$ be a subsequence of F such that $\{p_1 < p_2 < \ldots < p_k < \ldots\}$. Since $F = \{f_k\}_{k\in\mathbb{N}}$ is a sequence $st-$uniformly bounded, there exists a compact subset $\mathcal{P}$ of $C(X)$ such that $\{f_{p_k} : k \in \mathbb{N}\} \subset \mathcal{P}$, means that $f_{p_k} \in \mathcal{P}$ for each k. If the set Γ_F^u is empty then

$$\mathcal{P} \cap \Gamma_F^u = \emptyset.$$

Since $\{k \in \mathbb{N} : f_{p_k} \in \mathcal{P}\} \subset \{k \in \mathbb{N} : f_k \in \mathcal{P}\}$, by Lemma 4.2 and the final expression imply that the set $\delta\{k \in \mathbb{N} : f_{p_k} \in \mathcal{P}\} = 0$. This result contradicts with $\delta\{k \in \mathbb{N} : f_{p_k} \in \mathcal{P}\} > 0$. Then the set Γ_F^u is a nonempty.
For the second part of the Theorem, assume that $F = \{f_k\}_{k\in\mathbb{N}}$ is a $st-$ uniformly bounded sequence. By the Definition 4.1 there exists a compact subset $\mathcal{P}$ of $C(X)$ such that

$$\delta\{k \in \mathbb{N} : f_k \notin \mathcal{P}\} = 0$$

and the set Γ_F^u is nonempty. On the other hand

$$F = \{f_k\}_{k\in\mathbb{N}} = \{k \in \mathbb{N} : f_k \notin \mathcal{P}\} \cup \{k \in \mathbb{N} : f_k \in \mathcal{P}\}.$$

Now it is enough to show that $\Gamma_F^u \subset \mathcal{P}$. On the contrary, suppose that there exists an $\tilde{f} \in \Gamma_F^u$ such that $\tilde{f} \notin \mathcal{P}$. As $\mathcal{P}$ is compact there exists a number $\epsilon > 0$ such that $\{k \in \mathbb{N} : \left\|f_k - \tilde{f}\right\| < \epsilon\} \subset \{k \in \mathbb{N} : f_k \notin \mathcal{P}\}$ and therefore the set $\delta\{k \in \mathbb{N} : \left\|f_k - \tilde{f}\right\| < \epsilon\} = 0$. It is contrary to the assumption $\tilde{f} \in \Gamma_F^u$. Then the set Γ_F^u is compact. □

5. Γ-st-uniform convergence to compact set

Let the sequence of functions $F = \{f_k\}_{k\in\mathbb{N}}$ in $C(X)$ and $\mathcal{K} \subset C(X)$ be a nonempty compact set satisfying

$$\delta\{k \in \mathbb{N} : \rho(\mathcal{K}, f_k) \geq \epsilon\} = 0$$

holds for every $\epsilon > 0$. It is clear that $\mathcal{K} \subset \mathcal{B}_\epsilon(\mathcal{K})$ for every $\epsilon > 0$ and the set $\mathcal{B}_\epsilon(\mathcal{K})$ contains almost all functions of $F = \{f_k\}_{k\in\mathbb{N}}$. If there exists an $\epsilon_1 > 0$ such that

$$\delta\{k \in \mathbb{N} : \rho(\mathcal{K}_0, f_k) \geq \epsilon_1\} \neq 0$$

for each compact set $\mathcal{K}_0 \subset \mathcal{K}$ such that $\mathcal{K} \setminus \mathcal{K}_0 \neq \emptyset$ then we say that $\mathcal{K}$ is the smallest compact set.
Now we can define

Definition 5.1. *The sequence of functions $F = \{f_k\}_{k\in\mathbb{N}}$ in $C(X)$ is $\Gamma - st-$uniformly convergent to the compact set $\mathcal{K}$. In this case,*

$$\delta\{k \in \mathbb{N} : \rho(\mathcal{K}, f_k) \geq \epsilon\} = \delta\{k \in \mathbb{N} : \inf_{g\in\mathcal{K}} \|f_k - g\| \geq \epsilon\} = 0 \tag{3}$$

for every $\epsilon > 0$. We will call the compact set $\mathcal{K}$ the Γ- st limit of $F = \{f_k\}_{k\in\mathbb{N}}$. It is clear that if $F = \{f_k\}_{k\in\mathbb{N}}$ is Γ-st- uniformly convergent to a singleton $\{f\}$, then we have the sequence F is st- uniformly convergent to the function $\{f\}$.

Now we have another

Theorem 5.1. *If $F = \{f_k\}_{k\in\mathbb{N}}$ be a sequence st$-$ uniformly bounded of functions in $C(X)$. The smallest nonempty closed set $\mathcal{K}$ which holds Equation 3 is Γ_F^u that is $\mathcal{K} = \Gamma_F^u$.*

Proof. We know from the Theorem 4.2 that $\Gamma_F^u \neq \emptyset$ and, from the Definition 5.1, the set Γ_F^u satisfies $\delta\{k : \rho(\Gamma_F^u, f_k) \geq \varepsilon\} = \delta\{k \in \mathbb{N} : \inf_{g\in\Gamma_F^u} \|f_k - g\| \geq \epsilon\} = 0$ for every $\epsilon > 0$. Now suppose that Γ_F^u is not minimal, then there exists a compact set $\mathcal{K} \subset \Gamma_F^u$ such that $\Gamma_F^u \setminus \mathcal{K} \neq \emptyset$. In this case there exists a suc - function h, such that $h \notin \mathcal{K}$. Then there exists an $\epsilon > 0$ such that $\mathcal{B}_\epsilon(h) \cap \mathcal{B}_\epsilon(\mathcal{K}) = \emptyset$. Since h is a suc - function we can write

$$\delta\{k \in \mathbb{N} : \quad \|f_k - h\| < \epsilon \} > 0$$

On the other hand, we have

$$\{k \in \mathbb{N} : \quad \|f_k - h\| < \epsilon\} \subset \{k : f_k \notin \mathcal{B}_\epsilon(\mathcal{K})\}.$$

Combining these results, we get $\delta\{k : f_k \notin \mathcal{B}_\epsilon(\mathcal{K})\} > 0$, which contradicts with Definition 5.1. This completes the proof of the Theorem. □

Corollary 5.1. *Let Γ_F^u be a set of suc - functions of the sequence $F = \{f_k\}_{k\in\mathbb{N}}$. If $\delta\{k : \rho(\Gamma_F^u, f_k) \geq \varepsilon\} = 0$ for every $\epsilon > 0$ then the sequence $F = \{f_k\}_{k\in\mathbb{N}}$ is Γ-st uniformly convergent to the set Γ_F^u.*

Remark 5.1. If we take the following sequence of functions in $C([0,1])$ define by $f_k(x) = \begin{cases} k & \text{if } k \text{ is even} \\ 1 & \text{if } k \text{ is odd} \end{cases}$ for every $x \in [0,1]$ and $k \in \mathbb{N}$. It is clear that the sequence of functions is not uniformly bounded. Let $\mathcal{K} = \{f_k : k \text{ is even}\}$. The set $\mathcal{K}$ is uniformly unbounded closed equicontinuous. Then the set $\mathcal{K}$ is not compact. Hence we have

$$\{k \in \mathbb{N} : f_k \in \mathcal{K}\} = \{2, 4, 6, \ldots\}$$

and the set of suc - functions Γ_F^u is $\{\underline{1}\}$. $\{\underline{1}\}$ means that $f(x) = 1$ for every $x \in [0,1]$. In this case we have $\mathcal{K} \cap \Gamma_F^u = \emptyset$ but $\delta\{k \in \mathbb{N} : f_k \in \mathcal{K}\} = \delta\{2, 4, 6, \ldots\} = \frac{1}{2} \neq 0$ i.e. $\delta\{2, 4, 6, \ldots\} \neq 0$. Then we get the set of suc- functions Γ_F^u is $\{\underline{1}\}$ but,

$$\delta\{k \in \mathbb{N} : \rho(\Gamma_F^u, f_k) \geq \epsilon\} = \delta\{k \in \mathbb{N} : \inf_{g\in\Gamma_F^u} \|f_k - g\| \geq \epsilon\} = 1/2$$

for every $\epsilon > 0$. Hence we can conclude that if $F = \{f_k\}_{k\in\mathbb{N}}$ is not a sequence st $-$uniformly bounded of functions defined in $C(X)$ then the Theorem 5.1 need not be true.

Theorem 5.2. *Let $F = \{f_k\}_{k\in\mathbb{N}}$ be a sequence st- uniformly bounded of functions defined in $C(X)$. Then there exists a sequence $H = \{h_k\}_{k\in\mathbb{N}}$ of functions in $C(X)$ such that*
(i) $F \equiv_{st} H$ that is:

$$\delta\{k : f_k \neq h_k\} = 0,$$

(ii) $\Gamma_F^u = L_H^u$.

Proof. Let $\varepsilon_q = 1/2^q$. Then from Theorem 5.1 for every q we have

$$\delta\{k \in \mathbb{N} : \rho(\Gamma_F^u, f_k) \geq 1/2^q\} = 0.$$

For a given q we choose a number S_q such that $\delta\{k \in \mathbb{N} : \rho(\Gamma_F^u, f_k) \geq 1/2^q\} < 1/q$ for every $k > S_q$. By Theorem 4.2 the set Γ_F^u is nonempty and so we can take any function $f \in \Gamma_F^u$. Define by a sequence $H = \{h_k\}_{k\in\mathbb{N}}$ for every $k \in \mathbb{N}$, $\quad S_q < k \leq S_{q+1}$,

$$h_k = \begin{cases} f & \text{if } \rho(\Gamma_F^u, f_k) \geq 1/2^{q+1} \\ \\ f_k & \text{if } \rho(\Gamma_F^u, f_k) < 1/2^{q+1} \end{cases},$$

We have

(a) if $1 \leq k \leq S_1$,

$$\{k \leq n \,:\, f_k \neq h_k, \} \subseteq \{k \leq n \,:\, \rho(\Gamma_F^u, f_k) \geq 1/2\}.$$

(b) if $S_1 < k \leq S_2$,

$$\{k \leq n \,:\, f_k \neq h_k\} \subseteq \{k \leq S_1 \,:\, f_k \neq h_k\} \cup \{S_1 < k \leq n \,:\, f_k \neq h_k\} \subseteq$$
$$\{k \leq S_1 \,:\, \rho(\Gamma_F^u, f_k) \geq 1/2\} \cup \{S_1 < k \leq n \,:\, \rho(\Gamma_F^u, f_k) \geq 1/4\} \subseteq$$
$$\{k \leq S_1 \,:\, \rho(\Gamma_F^u, f_k) \geq 1/4\} \cup \{S_1 < k \leq n \,:\, \rho(\Gamma_F^u, f_k) \geq 1/4\} =$$
$$\{k \leq n \,:\, \rho(\Gamma_F^u, f_k) \geq 1/4\}$$

Consequently;

$$\delta\{k \leq n \,:\, f_k \neq h_k\} \leq \delta\{k \leq n \,:\, \rho(\Gamma_F^u, f_k) \geq 1/4\} < 1/2$$

In general for $S_{q-1} < k \leq S_q$, we have $\delta\{k \leq n \,:\, f_k \neq h_k\} \leq \delta\{k \leq n \,:\, \rho(\Gamma_F^u, f_k) \geq 1/2^q\} < 1/q$. Then $\delta\{k \leq n \,:\, f_k \neq h_k\} = 0$ as, $n \to \infty$. Thus $F \equiv_{st} H$.
Now we show that $\Gamma_F^u = \Gamma_H^u$. Let $f \in \Gamma_F^u$ then for every $\varepsilon > 0$

$$\delta\{k \leq n \,:\, \|f_k - f\| < \varepsilon\} > 0.$$

Clearly
$\{k \in \mathbb{N} \,:\, \|f_k - f\| < \varepsilon\} = \{k \in \mathbb{N} \,:\, f_k = h_k \text{ and } \|f_k - f\| < \varepsilon\} \cup \{k \in \mathbb{N} \,:\, f_k \neq h_k \text{ and } \|f_k - f\| < \varepsilon\} \subset \{k \in \mathbb{N} \,:\, \|h_k - f\| < \varepsilon\} \cup \{k \in \mathbb{N} \,:\, f_k \neq h_k\}$.
Thus

$$\{k \leq n \,:\, \|f_k - f\| < \varepsilon\} \subseteq \{k \leq n \,:\, \|h_k - f\| < \varepsilon\} \cup \{k \leq n \,:\, f_k \neq h_k\}.$$

From definition and $\delta\{k \leq n \,:\, f_k \neq h_k\} = 0$ we have $\delta\{k \in \mathbb{N} \,:\, \|h_k - f\| < \varepsilon\} > 0$ and so $f \in \Gamma_H^u$. It follows that $\Gamma_F^u \subset \Gamma_H^u$.
By analogy we can show that $\Gamma_H^u \subset \Gamma_F^u$ and therefore $\Gamma_F^u = \Gamma_H^u$.
At last by construction of the sequence $H = \{h_k\}_{k \in \mathbb{N}}$ we observe that $L_H^u = \Gamma_H^u$. This completes the proof. □

6. **Conclusions**

In this paper, we introduce a new type of convergence notion by using the notion of the uniform convergence of a sequence of functions, namely, Gamma - statistical convergence. This new notion is close to the set of limit points and the set of cluster points. The set of statistical cluster points turn out to be very useful and interesting tool in turnpike theory to study optimal paths. It has also been discussed in convex or non-convex optimal control problems in discrete systems. In classical theory of convergence, statistical convergence has a special place and these are also active research area. We introduce the concept of the compact set of statistical uniform cluster functions and we also give some properties of the compact set of statistical uniform cluster functions. We investigate the concept of new convergence using the compact set of statistical uniform cluster functions.

Furthermore, we propose a notion of a new type convergence for a sequence of functions which is the appropriate device to study of optimal paths and turnpike theory in continuous system. It is expected that the ideas and techniques of this paper may encourage further research.

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