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IRREDUCIBLE COVARIANT REPRESENTATIONS ASSOCIATED TO AN R-DISCRETE GROUPOID

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Unei perechi covariante pozitiv definite (T,ρ) *relativ la un grupoid r-discret G, i se poate asocia, printr-o teoremă de tip Stinespring, o reprezentare covarintă* $(U,\widetilde{\rho})$. *Vom stabili o condiţie necesară şi suficientă de ireductibilitate pentru aceasta din urmă.*

Let (T,ρ) *be a positive definite covariant pair with respect to an r-discrete groupoid G. Using a theorem of type Stinespring, we can associate to the pair* (T,ρ) *a covariant representation* $(U,\widetilde{\rho})$. *In this article, we provide a necessary and sufficient condition of irreducibility for the representation* $(U,\widetilde{\rho})$.

Keywords: r-discrete groupoid, positive definite covariant pair, covariant representation

1. Introduction

For a suitable amenable, r-discrete, principal groupoid G with the unit space G^0 and the semigroup $\Im$ of its compact and open G-sets, we define a **covariant representation of** $\Im$, to be a pair (T,ρ) , where

(i) ρ is a *-representation of $C_0(G^0)$ (= the C^*-algebra of complex-valued functions on G^0, that are continuous and vanishing at infinity) on a (complex and separable) Hilbert space H;

(ii) $T=\{T(s)\mid s\in\Im\}$ is a family of operators on H such that $\|T(s)\|\le 1$ and $T(s)T(t)=T(st)$, for $s,t\in\Im$;

(iii) $T(s)\rho(a)=\rho(a\circ s)T(s)$, for $a\in C_0(G^0)$ and $s\in\Im$ (see Obs. (vi)); and

(iv) $T(s)=\rho(\aleph_s)$, for $s\in\Im$ such that $s\subseteq G^0$ (where $\aleph_s$ denote the characteristic function of s).

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Definition If T is a function from $\Im$ to $B(H)$ and ρ is a *-representation of $C_0(G^0)$ on H, then (T,ρ) is a **positive definite covariant pair** , if the following conditions are satisfied:

(i) $T(s)\rho(a) = \rho(a \circ s)T(s)$, for $a \in C_0(G^0), s \in \Im$;

(ii) $T(s)T(t) = T(st)$, for $s,t \in \Im$;

(iii) $T(s) = \rho(\aleph_s)$, for $s \in \Im$ with $s \subseteq G^0$; and

(iv) for each finite collection of points $s_1,...,s_n \in \Im$, the operator matrix $(T(s_i^{-1}s_j))_{1\le i,j\le n}$ acting on $H \oplus ... \oplus H$, is non-negative (this means that, $\sum_{i,j} < T(s_i^{-1}s_j)\xi_j, \xi_i >_H \ge 0$, for any $(\xi_1,...,\xi_n) \in H \times ... \times H$).

Theorem Let (T,ρ) be a positive definite covariant pair with values in $B(H)$. Then there are a Hilbert space $\widetilde{H}$, a covariant representation $(U,\widetilde{\rho})$ of $\Im$ on $\widetilde{H}$ and a Hilbert space isomorphism V mapping H into $\widetilde{H}$ such that $\rho(a) = V^*\widetilde{\rho}(a)V$, for $a \in C_0(G^0)$ and $T(t) = V^*U(t)V$, for $t \in \Im$.

We shall show a necessary and sufficient condition of irreducibility for the covariant representation $(U,\widetilde{\rho})$ from the above theorem in terms of the pair (T,ρ).

2. Preliminaries

The definitions for the notions of: amenable, r-discrete principal groupoid, *G*- set , C^* -algebra associated with a locally compact groupoid, can be found in [1], [2], [3], [4] or [5].

Now, to understanding more easily the content of this paper, we shall present , first of all, the most important stages of the proof of the theorem from "Introduction".

Let $\widetilde{H}_0$ be the set of all functions $f : \Im \to H$, such that

(1) there is a compact set K_f (depending of f) such that $K_f \subseteq G$ and $f(t) = 0$, for $t \cap K_f$ = Ø ; and

(2) if $t, t_1 \in \Im$ such that $t_1 \subseteq t$, then $f(t_1) = \rho(\aleph_{d(t_1)})f(t)$ (where $d(t_1) = t_1^{-1}t_1$).

We define on $\widetilde{H}_0$ a sesquilinear functional $<\cdot,\cdot>$ by the formula

$$<f,g>=\sum <T(s^{-1}t)f(t),g(s)>_H,$$

where the sum is over any two finite sets $\{t_i\},\{s_j\}$ in $\mathfrak{I}$ such that $K_f \subseteq \cup t_i$, $K_g \subseteq \cup s_j$, and such that $t_i \cap t_j = s_k \cap s_l = \emptyset$, if $i \neq j$, $k \neq l$.

Let $N=\{f \in \widetilde{H}_0 \mid <f,f>=0\}$. The sesquilinear functional $<\cdot,\cdot>$ from above passes to an inner product on $\widetilde{H}_0 / N$. (For this inner product we keep the notation $<\cdot,\cdot>$.)

Let $\widetilde{H}$ be the completion of $\widetilde{H}_0 / N$ in the associated norm. Next, we define

$$V: H \to \widetilde{H} \quad \text{by} \quad V\xi = \widetilde{\xi} + N,$$

where $\widetilde{\xi}(t) = \rho(E(\aleph_{t^{-1}}))\xi$ (E being the conditional expectation from $C^*(G)$ to $C^*(G^0) = C_0(G^0)$; see [4], p.104). This operator V is an isometry.

Note that, for $f \in \widetilde{H}_0$ and $t \in \mathfrak{I}$, if we denote $f_t\ (s) = f(t^{-1}s),\ \forall s \in \mathfrak{I}$, we obtain $f_t \in \widetilde{H}_0$.

Now, for $q \in \mathfrak{I}$, the operator U on $\widetilde{H}_0 / N$ defined by $U(q)(f+N) = f_q + N, \forall f \in \widetilde{H}_0$ satisfies the properties $U(q^{-1}) = U(q)^*$ and $U(st) = U(s)U(t)$, for $q,s,t \in \mathfrak{I}$.

Finally, for $a \in C_0(G^0)$, we define $\widetilde{\rho}_0(a)$ on $\widetilde{H}_0$ by the formula $(\widetilde{\rho}_0(a)f)(t) = \rho(a \circ t^{-1})f(t)$. Denote $\widetilde{\rho}(a)(f+N) = \widetilde{\rho}_0(a)f + N$, $\widetilde{\rho}$ is a *-representation of $C_0(G^0)$ on $\widetilde{H}$.

The pair $(U,\widetilde{\rho})$ and the operator V have the properties asked in the theorem.

Observations (i) $(U,\widetilde{\rho})$ is just a positive definite covariant pair: let $(\xi_1,...,\xi_n) \in H \times ... \times H$ and $s_1,...,s_n \in \mathfrak{I}$; then

$$0 \le \sum_{i,j=1}^{n} \langle T(s_i^{-1}s_j)\xi_j,\xi_i\rangle = \sum_{i,j=1}^{n} \langle U(s_i^{-1}s_j)V\xi_j,V\xi_i\rangle \ .$$

From here, using $V(H) = \widetilde{H}$, we deduce that $(U(s_i^{-1}s_j))_{i,j=1}^{n}$ is a positive operator matrix.

(ii) As V is an isometry, it results $V^*V = I_H$. Since V is an isomorphism, there exists V^{-1} and $V^*V = V^{-1}V = I_H$, hence $(V^* - V^{-1})V = 0$. Then $VH = \widetilde{H}$ implies $V^* = V^{-1}$. Consequently, $T(t) = V^*U(t)V \Leftrightarrow VT(t)V^* = U(t)$, $\forall t \in \Im$.

(iii) If ρ is a non-degenerate representation of $C^*(G^0)$, then the family $\{T(s)\}_{s\in\Im}$ is non-degenerate (i.e. $\overline{\mathrm{Sp}}\{T(s)\xi \mid s \in \Im, \xi \in H\} = H$) and so is $\{U(s)\}_{s\in\Im}$ also: let $\eta \in H$ and $\{a_i\}_i \subseteq C_c(G^0), \{\xi_i\}_i \subseteq H$ such that $\eta = \lim_i \rho(a_i)\xi_i$; fix $\{s_i\}_i \subseteq \Im$ such that $\forall i : \operatorname{supp} a_i \subseteq s_i^{-1}s_i$. Then

$$\eta = \lim_i \rho(\aleph_{s_i^{-1}s_i} a_i)\xi_i = \lim_i \rho(\aleph_{s_i^{-1}s_i})\rho(a_i)\xi_i = \lim_i T(s_i^{-1}s_i)\rho(a_i)\xi_i .$$

For the second part of the assertion : since V is an isometry, V, V^* are continuous operators. Therefore,
$\widetilde{H} = V(H) = V(\overline{\mathbf{Sp}}(T(\Im)H)) \subseteq \overline{\mathbf{Sp}}(VT(\Im)(H)) = \overline{\mathbf{Sp}}(VT(\Im)V^*(\widetilde{H})) = \overline{\mathbf{Sp}}(U(\Im)\widetilde{H})$
.

(iv) If (T,ρ) is a positive definite covariant pair, then $T^*(s) = T(s^{-1})$, $\forall s \in \Im$: from the proof of the theorem, we know that $U^*(t) = U(t^{-1}), \forall t \in \Im$, hence $T^*(t) = V^*U(t^{-1})V = T(t^{-1}), \forall t \in \Im$.

(v) If (T,ρ) is a positive definite covariant pair, then $\| T(s) \| \le 1, \forall s \in \Im$: let $s \in \Im$; then we have $\| T(s) \|^2 = \| T(s)^* T(s) \| = \| T(s^{-1})T(s) \| = \| T(s^{-1}s) \| =$ $\| \rho(\aleph_{s^{-1}s}) \| \le \| \aleph_{s^{-1}s} \| = 1$.

(vi) For $a \in C_0(G^0)$ and $s \in \Im$, we have $a \circ s = \aleph_s a \aleph_{s^{-1}}$ (where s from the right side is a G-set, while s from the left side is a notation for the function $u \mapsto d(u \cdot s)$; $u \cdot s$ is the element $x \in s$ with $r(x) = u$), hence $a \circ s^{-1} \circ s =$ $(a \circ s^{-1}) \circ s = (\aleph_{s^{-1}} a \aleph_s) \circ s = (\aleph_s \aleph_{s^{-1}}) a (\aleph_s \aleph_{s^{-1}}) = \aleph_{ss^{-1}} a \aleph_{ss^{-1}} = a \circ ss^{-1}$. Now:
$\rho(a)T(s) = \rho(a)T(ss^{-1}s) = \rho(a)T(ss^{-1})T(s) = \rho(a)\rho(\aleph_{ss^{-1}})T(s) = \rho(a \circ (ss^{-1}))T(s) =$ $\rho(a \circ s^{-1} \circ s)T(s) = T(s)\rho(a \circ s^{-1})$.

3. Irreducible representations of $C_0(G^0)$

Lema 1 If $(T_1,\rho_1),(T_2,\rho_2)$ are two positive definite covariant pairs of $\Im$ on $B(H)$ with ρ_1,ρ_2 non-degenerate representations of $C_0(G^0)$ such that

$$\forall n \in N^* \text{ and } \forall s_1,\ldots,s_n \in \Im:\ (T_1(s_i^{-1}s_j))_n \le (T_2(s_i^{-1}s_j))_n ,$$

while $(U_1, \widetilde{\rho}_1, \widetilde{H}_1)$, respectively $(U_2, \widetilde{\rho}_2, \widetilde{H}_2)$ are the corresponding covariant representations, then there exists a contraction $W \in B(\widetilde{H}_2, \widetilde{H}_1)$ (i.e. $\| W \| \leq 1$) satisfying:

(i) $WV_2 = V_1$;

(ii) $WU_2(s) = U_1(s)W, \forall s \in \Im$; and

(iii)

$W\widetilde{\rho}_2(a) = \widetilde{\rho}_1(a)W, \forall a \in C_0(G^0) \Leftrightarrow T_1(t)\rho_2(a \circ t^{-1}) = T_1(t)\rho_1(a \circ t^{-1}), \forall t \in \Im$, $\forall a \in C_0(G^0)$.

Proof: (i) Let $h_1,...,h_n \in H$ and $s_1,...,s_n \in \Im$. Then:

$$\| \sum_j U_1(s_j)V_1 h_j \|^2 = \sum_{i,j} \langle V_1^* U_1(s_i^{-1}s_j)V_1 h_j, h_i \rangle = \sum_{i,j} \langle T_1(s_i^{-1}s_j)h_j, h_i \rangle \leq \sum_{i,j} \langle T_2(s_i^{-1}s_j)h_j, h_i \rangle =$$

$$= \| \sum_j U_2(s_j)V_2 h_j \|^2 .$$

Defining $W : \widetilde{H}_2 \to \widetilde{H}_1$ by $WU_2(t)V_2 h = U_1(t)V_1 h$, we remark that W is a contraction with $WV_2 = V_1$.

(ii) For $s,t \in \Im$ and $h \in H$, we have

$$WU_2(s)U_2(t)V_2 h = WU_2(st)V_2 h = U_1(st)V_1 h = U_1(s)U_1(t)V_1 h = U_1(s)WU_2(t)V_2 h .$$

(iii) If $a \in C_0(G^0), t \in \Im$ and $h \in H$, it follows

$$W\widetilde{\rho}_2(a)U_2(t)V_2 h = WU_2(t)\widetilde{\rho}_2(a \circ t^{-1})V_2 h = WU_2(t)V_2\rho_2(a \circ t^{-1})h = U_1(t)V_1\rho_2(a \circ t^{-1})h$$

On the other hand,

$$\widetilde{\rho}_1(a)WU_2(t)V_2 h = \widetilde{\rho}_1(a)U_1(t)V_1 h = U_1(t)\widetilde{\rho}_1(a \circ t^{-1})V_1 h = U_1(t)V_1\rho_1(a \circ t^{-1})h .$$

Hence, $W\widetilde{\rho}_2(a) = \widetilde{\rho}_1(a)W \Leftrightarrow V_1^* U_1(t)V_1\rho_2(a \circ t^{-1})h = V_1^* U_1(t)V_1\rho_1(a \circ t^{-1})h \Leftrightarrow$ $T_1(t)\rho_2(a \circ t^{-1})h = T_1(t)\rho_1(a \circ t^{-1})h$ ■

Definition 2 Let H be a Hilbert space and $U \in B(H)$. U is a **positive operator** (and we shall denote this by $U \geq 0$), if $U = U^*$ and $\langle U(x), x \rangle \geq 0, \forall x \in H$.

Notation 3 For a Hilbert space H and a set M such that $M \subseteq B(H)$, we write M' for the **commutant** of M, i.e.

$$M' = \{U \in B(H) \mid \forall V \in M : UV = VU\}$$

Lemma 4 Let (T,ρ) a covariant positive definite pair on $B(H)$ with ρ a non-degenerate representation of $C_0(G^0)$, $(U,\widetilde{\rho})$ the corresponding covariant representation on $B(\widetilde{H})$, and V the isomorphism between H and $\widetilde{H}$ such that

$$\rho(a)=V^*\widetilde{\rho}(a)V\ ,\forall a\in C_0(G^0) \text{ and } T(t)=V^*U(t)V\ ,\ \forall t\in\Im .$$

For $W\in U(\Im)'$ and $S\in\widetilde{\rho}(C_0(G^0))'$, we define the applications

$$\Phi_W:\Im\to B(H) \text{ by } \Phi_W(t)=V^*WU(t)V\ ,\ \forall t\in\Im \text{ and}$$

$$\Psi_S:C_0(G^0)\to B(H) \text{ by } \Psi_S(a)=V^*S\widetilde{\rho}(a)V\ ,\ \forall a\in C_0(G^0).$$

Under these conditions, the following assertions are true:

(i) the maps $W\mapsto\Phi_W$ and $S\mapsto\Psi_S$ are linear and injective;

(ii) if $W\in U(\Im)'$ and $0\le W$, then for $n\in N^*$ and $s_1,...,s_n\in\Im$, the operator matrix $(\Phi_W(s_i^{-1}s_j))_{i,j=1}^n$ is non-negative; and

(iii) if $W\in U(\Im)'$ such that $W=W^*$ and $\forall n\in N^*,\forall s_1,...,s_n\in\Im$, the operator matrix $(\Phi_W(s_i^{-1}s_j))_{i,j=1}^n$ is non-negative, then $W\ge 0$.

Proof: (i) It suffices to show, that the linear map $W\mapsto\Phi_W$ is injective. For the injectivity of $S\mapsto\Psi_S$, the proof is analogous.

We assume that $\Phi_W=0$. Let $s,t\in\Im$ and $h,k\in H$. Then:

$$\langle WU(t)Vh,U(s)Vk\rangle=\langle U(s^{-1})WU(t)Vh,Vk\rangle=\langle V^*WU(s^{-1}t)Vh,k\rangle=\langle\Phi_W(s^{-1}t)h,k\rangle=0$$

hence, $W=0$. (One applies Obs. (iii)).

(ii) We use the observation (i) from "Preliminaries" and the following theorem:

" If A is an involutive algebra, H a Hilbert space, π a representation of A on H and $T\in B(H)$ such that $T\ge 0$ and $T\in\pi(A)'$, then there exists $K\in B(H)$ with $K=K^*, K^2=T$ and $K\in\pi(A)'$. "

Thus, for $W\in U(\Im)'$ with $0\le W$, there is $K\in B(\widetilde{H})$ such that $K=K^*$, $K^2=W$ and $K\in U(\Im)'$.

Let $s_1,...,s_n\in\Im$ and $(\xi_1,...,\xi_n)\in H\oplus...\oplus H$. The required conclusion follows by

$$\sum_{i,j}\langle\Phi_W(\ s_i^{-1}s_j)\xi_j,\xi_i\rangle=\sum_{i,j}\langle V^*WU(s_i^{-1}s_j)V\xi_j,\xi_i\rangle=\sum_{i,j}\langle K^2U(s_i^{-1}s_j)V\xi_j,V\xi_i\rangle=$$

$$=\sum_{i,j}\langle U(s_i^{-1}s_j)KV\xi_j,KV\xi_i\rangle\ge 0$$.(One uses obs. (iii).)

(iii) Let W as in the statement of the lemma. For $n\in N^*$, $s_1,...,s_n\in\Im$, $h_1,...,h_n\in H$ and $k=U(s_1)Vh_1+...+U(s_n)Vh_n$, we get

$$\langle Wk,k\rangle = \sum_{i,j}\langle V^*U(s_i^{-1})WU(s_j)Vh_j,h_i\rangle = \sum_{i,j}\langle V^*WU(s_i^{-1}s_j)Vh_j,h_i\rangle =$$
$$= \sum_{i,j}\langle \Phi_W(s_i^{-1}s_j)h_j,h_i\rangle \geq 0,$$

hence, $W \geq 0$ ■

Lemma 5 Let (T_1,ρ_1) and (T,ρ) two positive definite covariant pairs with ρ_1,ρ non-degenerate representations of $C_0(G^0)$ such that

$$\forall n \in N^*, \forall s_1,...,s_n \in \Im:\ (T_1(s_i^{-1}s_j))_{i,j=1}^n \leq (T(s_i^{-1}s_j))_{i,j=1}^n \text{ and}$$
$$\forall t \in \Im, \forall a \in C_0(G^0):\ T_1(t)\rho_1(a \circ t^{-1}) = T_1(t)\rho(a \circ t^{-1}) .$$

If $(U,\widetilde{\rho})$ is the covariant representation with respect to (T,ρ), then there is $W \in U(\Im)' \cap \widetilde{\rho}(C_0(G^0))'$ with the properties: $0 \leq W \leq I$ ($I:\widetilde{H} \to \widetilde{H}$ is the identity operator, $I(h) = h, \forall h \in \widetilde{H}$) , $T_1 = \Phi_W$ and $\rho_1 = \Psi_W$.

Proof: Let $V: H \to \widetilde{H}$ the isomorphism corresponding to the pair (T,ρ), (U_1,ρ_1) the covariant representation and $V_1: H \to \widetilde{H}_1$ the isomorphism associated with the pair (T_1,ρ_1).

The lemma 1 assures the existence of a contraction $\widetilde{W}:\widetilde{H} \to \widetilde{H}_1$ such that $\widetilde{W}V = V_1$, $\widetilde{W}U(s) = U_1(s)\widetilde{W}$, $\forall s \in \Im$ and $\widetilde{W}\widetilde{\rho}(a) = \widetilde{\rho}_1(a)\widetilde{W}$, $\forall a \in C_0(G^0)$.

We shall denote $W = \widetilde{W}^*\widetilde{W}$. Then $W \geq 0$ and for $h \in \widetilde{H}$, we have: $\langle Wh,h\rangle = \langle \widetilde{W}h,\widetilde{W}h\rangle \leq \langle h,h\rangle$, hence $0 \leq W \leq I$.

Choose $s \in \Im$. Then,

$$WU(s) = \widetilde{W}^*U_1(s)\widetilde{W} = (U_1^*(s)\widetilde{W})^*\widetilde{W} = (U_1(s^{-1})\widetilde{W})^*\widetilde{W} = (\widetilde{W}U(s^{-1}))^*\widetilde{W} =$$
$$= U(s^{-1})^*\widetilde{W}^*\widetilde{W} = U(s)W$$

implies $W \in U(\Im)'$. Analogously, $W \in \widetilde{\rho}(C_0(G^0))'$.

Suppose that $t \in \Im$ and $h,k \in H$. Since

$$\langle \Phi_W(t)h,k\rangle = \langle WU(t)Vh,Vk\rangle = \langle \widetilde{W}U(t)Vh,\widetilde{W}Vk\rangle = \langle U_1(t)\widetilde{W}Vh,\widetilde{W}Vk\rangle =$$
$$= \langle U_1(t)V_1h,V_1k\rangle = \langle V_1^*U_1(t)V_1h,k\rangle = \langle T_1(t)h,k\rangle ,$$

$T_1 = \Phi_W$. Similarly, $\rho_1 = \Psi_W$ ■

Observations 6 (i) If s is a G-set, then $ss^{-1}, s^{-1}s$ are also G-sets and at the same time, they are subsets of G^0 . Moreover, $s = ss^{-1}s$.(If $x \in ss^{-1}s$, then $x = yu$, with $y \in s$ and $u \in s^{-1}s$, hence $x = y \in s$. Conversely, for $x \in s$, we have $x = xx^{-1}x \in ss^{-1}s$.)

(ii) The operator W from lemma 5 satisfies $W^2 = W$, as follows from

$$T_1 = \Phi_W \Rightarrow T_1(st) = \Phi_W(st), \forall s,t \in \Im \Rightarrow \Phi_W(s)\Phi_W(t) = T_1(s)T_1(t) = T_1(st) =$$
$$= \Phi_W(st) \Rightarrow V^*WU(s)VV^*WU(t)Vh = V^*WU(st)Vh, \forall s,t \in \Im, \forall h \in H \Rightarrow$$
$$V^*W^2U(st)Vh = V^*WU(st)Vh, \forall s,t \in \Im, \forall h \in H \Rightarrow$$
$$W^2U(st)Vh = WU(st)Vh, \forall s,t \in \Im, \forall h \in H \Rightarrow W^2U(ss^{-1}s)\widetilde{h} = WU(ss^{-1}s)\widetilde{h},$$
$$\forall s \in \Im, \forall \widetilde{h} \in \widetilde{H} \Rightarrow W^2U(s)\widetilde{h} = WU(s)\widetilde{h}, \forall s \in \Im, \forall \widetilde{h} \in \widetilde{H}.$$

Proposition 7 Let (T,ρ) a positive definite covariant pair. Then there is a bijection between the set

$$A = \{W \in U(\Im)' \cap \widetilde{\rho}(C_0(G^0))' \mid 0 \le W \le I, W^2 = W\}$$

and the family B of the positive definite covariant pairs (S,θ) on $B(H)$, which have the property

$$\forall n \in N^*, \forall s_1,...,s_n \in \Im: \ (S(s_i^{-1}s_j))_{i,j=1}^n \le (T(s_i^{-1}s_j))_{i,j=1}^n \text{ and}$$
$$S(t)\rho(a \circ t^{-1}) = S(t)\theta(a \circ t^{-1}), \forall a \in C_0(G^0), \forall t \in \Im.$$

Proof: Assume $W \in A$. By lemma 4, for $n \in N^*$ and $t_1,...,t_n \in \Im$, the operator matrix $(\Phi_W(t_i^{-1}t_j))_{i,j=1}^n$ is non-negative. Because $W \le I$, we can claim that $\forall n \in N^*$ and $\forall t_1,...,t_n \in \Im: (\Phi_{I-W}(t_i^{-1}t_j))_{i,j=1}^n$ is a non-negative matrix, hence $\Phi_W \le T$.

Using $W^2 = W$, it is clear that $\Phi_W(st) = \Phi_W(s)\Phi_W(t), \forall s,t \in \Im$ and that Ψ_W is a representation of $C_0(G^0)$.

For $s \in \Im$ such that $s \subseteq G^0$, we have

$\Phi_W(s) = V^*WU(s)V = V^*W\widetilde{\rho}(\aleph_s)V =$

$\Psi_W(\aleph_s)$. Next, for $a \in C_0(G^0)$ and $s \in \Im$, it follows

$$\Phi_W(s)\Psi_W(a) = V^*WU(s)VV^*W\widetilde{\rho}(a)V = V^*W^2U(s)\widetilde{\rho}(a)V =$$
$$= V^*W^2\widetilde{\rho}(a \circ s)U(s)V = V^*W\widetilde{\rho}(a \circ s)VV^*WU(s)V = \Psi_W(a \circ s)\Phi_W(s).$$

Finally, if $t \in \Im$ and $a \in C_0(G^0)$, then

$$\Phi_W(t)\rho(a \circ t^{-1}) = V^*WU(t)V\rho(a \circ t^{-1}) \text{ and}$$
$$\Phi_W(t)\Psi_W(a \circ t^{-1}) = V^*WU(t)V\ V^*W\widetilde{\rho}(a \circ t^{-1})V = V^*WU(t)\widetilde{\rho}(a \circ t^{-1})V =$$
$$= V^*WU(t)V\rho(a \circ t^{-1}).$$

Thus, $\Phi_W(t)\rho(a \circ t^{-1}) = \Phi_W(t)\Psi_W(a \circ t^{-1})$. All these imply $(\Phi_W, \Psi_W) \in B$.

By the lemma 4, the map $W \mapsto (\Phi_W, \Psi_W)$ is injective. Its surjectivity follows from lemma 5 and observation 6 (ii) ■

Proposition 8 If (T,ρ) is a non-null, positive definite covariant pair such that, for every $(S,\theta)\in B$, there is $\lambda\in C$ with $S=\lambda T$, then the covariant representation $(U,\widetilde{\rho})$ associated with (T,ρ) is irreducible (in the sense that, only the subspaces 0 and $\widetilde{H}$ are closed and invariant with respect to $U(\Im)$ and $\widetilde{\rho}(C_0(G^0))$

Proof: Let (T,ρ) and $(S,\theta)\in B$ as in the statement of the proposition. By lemma 5, we can find $W\in A$ such that $S=\Phi_W$ and $\theta=\Psi_W$. Consequently, there is
$\lambda\in C$ such that $\Phi_W=\lambda T$. It results:

$$V^*WU(t)V=\lambda T(t),\forall t\in\Im\Leftrightarrow WU(t)=\lambda VT(t)V^*,\forall t\in\Im\Leftrightarrow$$
$$WU(t)=\lambda U(t),\forall t\in\Im\Leftrightarrow W=\lambda I$$

Let K a closed subspace of $\widetilde{H}$, such that K is invariant with respect to $U(\Im)$ and $\widetilde{\rho}(C_0(G^0))$. For $h\in\widetilde{H}$, we consider the writting $h=h_1+h_2$, where $h_1\in K$ and $h_2\in K^{\perp}$. We shall denote with P_K, the projection on K, $P_K:\widetilde{H}\to\widetilde{H}$, $P_Kh=h_1$. It remains to show that, $P_K\in A$ (A from proposition 7). Then it will exist $\lambda\in C$ such that $P_K=\lambda I$. This is equivalent with $K=0$ or $K=\widetilde{H}$.

For $k\in K,h\in K^{\perp}$ and $s\in\Im$, we have $U(s^{-1})k\in K$, hence $\langle U(s)h,k\rangle=\langle h,U(s^{-1})k\rangle=0$, from where $U(s)h\in K^{\perp}$. Thus, if $h\in\widetilde{H}$ such that $h=h_1+h_2$, with $h_1\in K$ and $h_2\in K^{\perp}$, we can write

$$P_KU(s)h=P_K(U(s)h_1+U(s)h_2=U(s)h_1=U(s)P_Kh,$$

i.e. $P_K\in U(\Im)'$. Analogously, it results that $P_K\in\widetilde{\rho}(C_0(G^0))'$.

For $h\in\widetilde{H}$ with the same decomposition as above, we find

$$P_K^2h=P_Kh_1=h_1=P_Kh,\ \langle P_Kh,h\rangle=\langle h_1,h_1+h_2\rangle=\langle h_1,h_1\rangle\geq 0 \text{ and}$$
$$\langle (P_K-I)h,h\rangle=\langle h_1-h,h\rangle=-\langle h_2,h\rangle=-\langle h_2,h_2\rangle\leq 0.$$

Hence $P_K^2=P_K$ and $0\leq P_K\leq I$ ■

Proposition 9 If (T,ρ) is a non-null, positive definite covariant pair such that ρ is a non-degenerate representation of $C_0(G^0)$ and $(U,\widetilde{\rho})$ is irreducible, then for $\forall(S,\theta)\in B$, there is $\lambda\in C$ such that $S=\lambda T$.

Proof: Choose (T,ρ) and $(U,\widetilde{\rho})$ with the properties from the statement and $(S,\theta)\in B$. By lemma 5, there is an operator $W\in A$ such that $(S,\theta)=(\Phi_W,\Psi_W)$.

Let $K = W(\widetilde{H})$. Then

$$W(\widetilde{H}) = \{h \in \widetilde{H} \mid \exists\ k \in \widetilde{H} \text{ a.î. } h = Wk\} = \{h \in \widetilde{H} \mid h = Wk\}$$

is a linear closed subspace of $\widetilde{H}$. Moreover, it is invariant with respect to $U(\Im)$ and $\widetilde{\rho}(C_0(G^0))$. Because $(U, \widetilde{\rho})$ is irreducible, we deduce that $W(\widetilde{H})$ is 0 or $\widetilde{H}$, hence $W = \lambda I$, for some $\lambda \in C$. Finally,
$\forall t \in \Im : S(t) = \Phi_W(t) = V^* \lambda U(t) V = \lambda T(t)$ ■

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