COMPOSITE ALGORITHM FOR SOLVING SPLIT VARIATIONAL INEQUALITY PROBLEM AND SPLIT VARIATIONAL INCLUSION PROBLEMS

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This paper aims to investigate a new composite algorithm for solving a split variational inequality problem and a split variational inclusion problem in real Hilbert spaces. Under very mild conditions, we prove a strong convergence theorem for the proposed algorithm. Furthermore, an application is given to illustrate the effectiveness of the algorithm. The results improve and extend the corresponding ones announced by some others in the earlier and recent literature.

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1. Introduction

Throughout this paper, let H be a real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$, and C be a nonempty closed convex subset of H. Fix(T) is denoted as the set of fixed points of a nonlinear mapping T. We use $x_n \rightarrow x$ and $x_n \rightharpoonup x$ to indicate the strong convergence and the weak convergence of the sequence$\{x_n\}$ to x, respectively.

First, we recall some notations which are needed in sequel. A mapping $T : H \rightarrow H$ is called:

(a) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H;$$

(b) firmly nonexpansive if

$$\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2, \quad \forall x, y \in H;$$

(c) α-inverse strongly monotone if there exists a positive constant α such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2, \quad \forall x, y \in H.$$

A multi-valued mapping $B : D(B) \subseteq H \rightarrow 2^H$ is called monotone if, for all $x, y \in D(B)$, $u\in Bx$ and $v\in By$ such that

$$\langle x-y,\ u-v\rangle\geq 0.$$

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For every point $x \in H$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x-P_C x\| \leq \|x-y\|, \quad \forall y \in C.$$

P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive mapping and satisfies the following inequalities:

$$\langle x-P_C x, y-P_C x\rangle \leq 0, \quad \forall x \in H, y \in C;$$
$$\|P_C x-P_C y\|^2 \leq \langle x-y, P_C x-P_C y\rangle, \quad \forall x, y \in H;$$
$$\|P_C x-y\|^2 \leq \|x-y\|^2-\|x-P_C x\|^2, \quad \forall x \in H, y \in C.$$

Let $V : C \to H$ be a single-valued continuous self-mapping, the variational inequality problem (VIP) is to find $u \in C$ such that

$$\langle Vu, v-u \rangle \geq 0, \quad \forall v \in C,$$

the set of solutions of the VIP (1) is denoted by $VI\,(C, V)$.

Variational inequality theory has emerged as a great important tool in studying a wide class of unilateral, obstacle, equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. Subsequently, Censor et al. [1] introduced the following split variational inequality problem (SVIP): find a point x^*, satisfying

$$x^* \in VI\,(C_1, f) \text{ and } Ax^* \in VI\,(C_2, g), \tag{1}$$

where C_i is nonempty closed convex of H_i (i=1, 2), f, g are two inverse strongly monotone mappings and A is a bounded linear operator of H_1 into H_2. For solving SVIP (1), they introduced the following algorithm:

$$x_{n+1} = P_{C_1}(I-\lambda f)(x_n+\gamma A^*(P_{C_2}(I-\lambda g)-I)Ax_n), \quad \forall n \in \mathbb{N}, \tag{2}$$

under some mild restrictions on the parameters, they proved that the sequence $\{x_n\}$ generated by (2) converges weakly to a solution of SVIP (1).

A monotone mapping B is maximal if the Graph(B) is not properly contained in the graph of any other monotone mapping. It is well known that a monotone mapping B is maximal if and only if for $(x, u) \in D(B) \times H, \langle x - y, u - v\rangle \geq 0$ for every(y, v)∈Graph(B) implies that $u \in Bx$.

Let $B : Dom(B) \subseteq H \to 2^H$ be a multi-valued maximal monotone mapping. For some $\lambda > 0$, the resolvent operator $J_\lambda^B : H_1 \to Dom(B)$ associated with B is defined by

$$J_\lambda^B x := (I + \lambda B)^{-1}(x), \quad \forall x \in H,$$

where I stands for the identity operator on H_1. Observe that for all $\lambda>0$, the resolvent operator J_λ^B is single-valued, nonexpansive and firmly nonexpansive.

Split variational inclusion problem has already been used in practice as a model in intensity-modulated radiation therapy treatment planning; see e.g., [2, 3]. This formalism is also at the core of the modeling of many inverse problems arising for phase retrieval and other real-world problems, further, in sensor networks in computerized tomography and data compression; see e.g., [4, 5] and references therein.

In 2011, Moudafi [6] introduced the following split monotone variational inclusion problem (in short, SMVIP): find $x^* \in H_1$ such that

$$\begin{cases} 0\in f_1x^*+B_1x^*, \\ y^* = Ax^* \in H_2 : 0 \in f_2y^* + B_2y^*, \end{cases} \tag{3}$$

where $f_1: H_1 \to H_1$ and $f_2: H_2 \to H_2$ are two given single-valued mappings, $A: H_1 \to H_2$ is a bounded linear operator, $B_1: H_1 \to 2^{H_1}$ and $B_2: H_2 \to 2^{H_2}$ are two multi-valued maximal monotone mappings.

If $f_1 \equiv 0, f_2 \equiv 0$, then the problem (3) reduces to the following split variational inclusion problem: find $x^* \in H_1$ such that

$$\begin{cases} 0\in B_1x^*, \\ y^* = Ax^* \in H_2 : 0 \in B_2y^*. \end{cases} \tag{4}$$

Subsequently, Byrne et al. [7] proved a weak and strong convergence of the following iterative method for problem (4): for given $x_0 \in H_1$ and $\lambda > 0$, compute the following iterative sequence:

$$x_{n+1} = J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n].$$

Recently, Sumalai et al. [8] studied a new split variational inclusion problem:

$$\begin{cases} 0\in Kx^*, \\ L_\alpha x^* \in H_\alpha : 0 \in K_\alpha(L_\alpha x^*), \end{cases}$$

where $L_\alpha: H \to H_\alpha$ is bounded linear operator for every $\alpha = 1,2,\cdots,N$, $K: H \to 2^H$ and $K_\alpha: H_\alpha \to 2^{H_\alpha}$ are multi-valued maximal monotone mappings. Moreover, they introduced the following scheme:

$$z_{n+1} = J_{\mu_{n,\alpha}}^K[\vartheta_n v + (1-\vartheta_n)\sum_{\alpha=1}^N \varrho_{n,\alpha}(z_n - \eta_{n,\alpha}L_\alpha^*(I - J_{\mu_{n,\alpha}}^{K_\alpha})L_\alpha z_n)].$$

As a result, they proved a strong convergence of the algorithm above under appropriate assumptions on the parameters.

Let F: $C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for F is to find $z\in C$ such that

$$F(z, y) \geq 0, \quad \forall y \in C. \tag{5}$$

The set of all solutions of the problem (5) is denoted by $EP(F)$, i.e.,

$$EP(F) = \{z \in C : F(z, y) \geq 0, \forall y \in C\}.$$

Very recently, Zheng et al. [9] proposed a new composite split problem: find $x^* \in C$ such that

$$x^* \in VI(C, f_1) \cap EP(\psi),\ Ax^* \in VI(C, f_2) \cap EP(\varphi),$$

where f_1, f_2 are two inverse strongly monotone mapping and ψ, φ are two bifunction. In order to solve the split problem, they proposed the following algorith

$$x_{n+1} = P_C(I - \alpha_1 f_1) T_{\lambda_1}^{\varphi}[\gamma_n u + (1 - \gamma_n)(x_n - \beta A^*(Ax_n - P_Q(I - \alpha_2 f_2) T_{\lambda_2}^{\psi} Ax_n))].$$

As a result, they proved a strong convergence of the algorithm above under appropriate assumptions on the parameters.

Motivated and inspired by the results above, we introduce a new algorithm for solving problem (1) and problem (4). Under some suitable assumptions, a strong convergence of the proposed algorithm is proved for solving a split variational inclusion problem. Finally, we apply the main results to solve a split convex optimization problem.

2. Preliminaries

In this section, we first recall some lemmas which are needed in sequel.

Lemma 2.1. ([10]) *Let a multivalued mapping K: $Dom(K) \subseteq H \to 2^H$* be monotone, then for all $z \in Ran(I + \mu K)$ with $\mu > 0$, we have

$$\|J_\mu^K z - s\|^2 \leq \|z - s\|^2 - \|z - J_\mu^K z\|^2,$$

where $s \in \Gamma = K^{-1}(0) \neq \emptyset$.

Lemma 2.2. ([11]) *Let K be a nonexpansive mapping on a closed convex subset C of a real Hilbert space H. The mapping I-K is said to be demiclosed on C, if for any sequence z_n in C, such that $z_n \rightharpoonup s \in C$ and $(I - K)(z_n) \to s^*$, we have $(I - K)(s) = s^*$.*

Lemma 2.3. ([12]) *Let H_1 be a real Hilbert space. Then, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H_1.$$

Lemma 2.4. ([13]) *Let f: $C \to H_1$ be an inverse strongly monotone operator. Then*

$$u \in VI(C, f) \Leftrightarrow u = P_C(u - \lambda f u), \quad \forall \lambda > 0.$$

Lemma 2.5. ([14]) *Let f: $C \to H_1$ be an α-inverse strongly monotone mapping. Then, for all $x, y \in C$ and $\lambda > 0$, one has*

$$\|(I-\lambda f)x-(I-\lambda f)y\|^2 \leq \|x-y\|^2 + \lambda(\lambda-2\alpha)\|fx-fy\|^2.$$

In particular, if $\lambda \in (0, 2\alpha]$, $I-\lambda f$: $C \to H_1$ is a nonexpansive mapping.

Lemma 2.6. ([15]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation :*

$$a_{n+1} \leq (1-\theta_n)\, a_n + \sigma_n, \quad n \geq 0,$$

where $\{\theta_n\}$ is a sequence in (0, 1) and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \theta_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\sigma_n}{\theta_n} \leq 0$ or $\sum_{n=1}^{\infty} |\sigma_n| < \infty$.

Then, $\lim\limits_{n\to\infty} a_n = 0$.

3. Main results

In this section, we introduce a new algorithm to approximating a common solution of problem (1) and problem (4). Let Ω_1 and Ω_2 denote the solution set of problem (1) and problem (4), respectively. Subsequently, we give the main results of the following algorithm.

Algorithm 1 Let $u \in H_1$ be a fixed point and choose an arbitrary initial point $x_0 \in H_1$. For $n \in \mathbb{N}$, let $\{x_n\}$ be a sequence of H_1 generated by:

$$\begin{cases} z_n = x_n - \beta A^*[Ax_n - P_Q(I-\alpha_2 g)J_{\mu_2}^{B_2} Ax_n], \\ y_n = P_C(I-\alpha_1 f)J_{\mu_1}^{B_1}[\gamma_n u + (1-\gamma_n)z_n], \\ x_{n+1} = (1-\lambda_n x_n) + \lambda_n y_n. \end{cases} \tag{6}$$

Assume the following conditions hold:

(i) $0 < \alpha_1 \leq 2k$, $0 < \alpha_2 \leq 2l$;

(ii) $0 < \mu_1 < \infty$, $0 < \mu_2 < \infty$;

(iii) $0 < \beta < \frac{1}{\|A\|^2}$;

(iv) $0 < \gamma_n < \limsup_{n\to\infty} \gamma_n < 1$, $\lim\limits_{n\to\infty} \gamma_n = 0$, $\sum_{j=1}^{\infty} \gamma_n = \infty$, $\lim\limits_{n} \frac{\gamma_n - \gamma_{n-1}}{\gamma_n} = 0$;

(v) $0 < \liminf_{n\to\infty} \lambda_n < \limsup_{n\to\infty} \lambda_n < 1$.

Theorem 3.1. *Let H_i, i=1, 2 be real Hilbert spaces and C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Assume that $B_1, B_2: H_1 \to 2^{H_1}$ are two maximal monotone operators and f, g are k, l-inverse strongly monotone mappings, respectively. Let $A: H_1 \to H_2$ be bounded linear operators with adjoint operators A^*. Assume that $\Gamma = \Omega_1 \cap \Omega_2 \neq \emptyset$, conditions (i)-(v) hold. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to the point $p = P_\Gamma u$.*

Proof. First, we claim that the sequence $\{x_n\}$ generated by (6) is bounded. Indeed, take $p = P_\Gamma$, then $p \in VI(C, f)$, $Ap \in VI(Q, g)$, $p \in \mathrm{Fix}\,(J_{\mu_1}^{B_1})$, $Ap \in \mathrm{Fix}\,(J_{\mu_2}^{B_2})$.
Set $u_n = J_{\mu_2}^{B_2} Ax_n$, $v_n = P_Q(I - \alpha_2 g) u_n$, $w_n = \gamma_n u + (1 - \gamma_n) z_n$, $l_n = J_{\mu_1}^{B_1} w_n$. Since $P_Q(I - \alpha_2 g)$, $J_{\mu_2}^{B_2}$ is non-expansive and $J_{\mu_2}^{B_2}$ is firmly-nonexpansive, we deduce:

$$\|u_n - Ap\| = \|J_{\mu_2}^{B_2} Ax_n - J_{\mu_2}^{B_2} Ap\| \le \|Ax_n - Ap\|,$$
$$\|l_n - p\| = \|J_{\mu_1}^{B_1} w_n - J_{\mu_1}^{B_1} p\| \le \|w_n - p\|, \tag{7}$$
$$\begin{aligned}\|v_n - Ap\|^2 &= \|P_Q(I-\alpha_2 g) J_{\mu_2}^{B_2} Ax_n - P_Q(I-\alpha_2 g) J_{\mu_2}^{B_2} Ap\|^2 \\ &\le \|J_{\mu_2}^{B_2} Ax_n - J_{\mu_2}^{B_2} Ap\|^2 \\ &\le \|Ax_n - Ap\|^2 - \|u_n - Ax_n\|^2. \end{aligned}\tag{8}$$

From (6) and (7), we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-\lambda_n) x_n + \lambda_n y_n - p\| \\ &\le (1-\lambda_n)\|x_n - p\| + \lambda_n \|y_n - p\| \\ &\le (1-\lambda_n)\|x_n - p\| + \lambda_n \|P_C(I - \alpha_1 f) l_n - p\| \\ &\le (1-\lambda_n)\|x_n - p\| + \lambda_n \|l_n - p\| \\ &\le (1-\lambda_n)\|x_n - p\| + \lambda_n \|w_n - p\|. \end{aligned}\tag{9}$$

In addition, we observe that

$$\begin{aligned}\langle Ax_n - Ap, v_n - Ax_n\rangle &= \langle v_n - Ap, v_n - Ax_n\rangle - \|v_n - Ax_n\|^2 \\ &= \frac{1}{2}(\|v_n - Ap\|^2 + \|v_n - Ax_n\|^2 - \|Ax_n - Ap\|^2) - \|v_n - Ax_n\|^2,\end{aligned}$$

it follows from (8) that

$$\begin{aligned}\langle Ax_n - Ap, v_n - Ax_n\rangle &\le \frac{1}{2}(\|Ax_n - Ap\|^2 - \|u_n - Ax_n\|^2 + \|v_n - Ax_n\|^2 - \|Ax_n - Ap\|^2) \\ &= -\frac{1}{2}\|u_n - Ax_n\|^2 - \frac{1}{2}\|v_n - Ax_n\|^2. \end{aligned}\tag{10}$$

From (10), we have

$$\begin{aligned}\|z_n - p\|^2 &= \|x_n - \beta A^*(Ax_n - v_n) - p\|^2 \\ &= \|x_n - p\|^2 - 2\beta\langle Ax_n - Ap, Ax_n - v_n\rangle + \beta^2 \|A^*(Ax_n - v_n)\|^2 \\ &\le \|x_n - p\|^2 - \beta(\|u_n - Ax_n\|^2 + \|v_n - Ax_n\|^2) + \beta^2 \|A\|^2 \|Ax_n - v_n\|^2\end{aligned}$$

$$= \|x_n-p\|^2-\beta\|u_n-Ax_n\|^2-\beta(1-\beta\|A\|^2)\|Ax_n-v_n\|^2$$
$$\leq\|x_n-p\|^2. \tag{11}$$

By the definition of w_n, it follows from (11) that

$$\begin{aligned}\|w_n- p\| &= \|\gamma_n u + (1-\gamma_n) z_n- p\|\\ &\leq (1-\gamma_n)\|z_n- p\| + \gamma_n\|u - p\|\\ &\leq (1- \gamma_n)\|x_n- p\|+ \gamma_n\|u- p\|.\end{aligned} \tag{12}$$

In view of (9) and (12), we obtain

$$\begin{aligned}\|x_{n+1}-p\| &\leq (1-\lambda_n)\|x_n-p\|+\lambda_n\|w_n-p\|\\ &\leq (1-\lambda_n)\|x_n- p\| + \lambda_n[(1- \gamma_n)\|x_n- p\| + \gamma_n\|u - p\|]\\ &= (1-\lambda_n\gamma_n)\|x_n-p\|+\lambda_n\gamma_n\|u-p\|\\ &\leq \max \{ \|x_n-p\|, \|w_n-p\|\}\\ &\vdots\\ &\leq \max \{ \|x_0-p\|, \|w_n-p\|\},\end{aligned}$$

which implies that $\{x_n\}$ is bounded, and so are $\{u_n\},\{v_n\}$, $\{w_n\},\{l_n\}$ and $\{y_n\}$. From (6), we deduce

$$\begin{aligned}\|z_{n+1}- z_n\|^2 &= \|x_{n+1}- x_n- \beta[A^*(Ax_{n+1}- v_{n+1})-A^*(Ax_n- v_n)]\|^2\\ &= \|x_{n+1}-x_n\|^2-2\beta\langle Ax_{n+1}- Ax_n , Ax_{n+1}-Ax_n- (v_{n+1}-v_n) \rangle\\ &\quad + \beta^2\|A^*(Ax_{n+1}- v_{n+1})-A^*(Ax_n- v_n)\|^2\\ &\leq \|x_{n+1}-x_n\|^2+2\beta\langle Ax_{n+1}- Ax_n , v_{n+1}-v_n-(Ax_{n+1}- Ax_n)\rangle\\ &\quad +\beta^2\|A\|^2\|v_{n+1}- v_n- (Ax_{n+1}- Ax_n)\|^2\\ &= \|x_{n+1}- x_n\|^2+\beta^2\|A\|^2\|v_{n+1}-v_n-(Ax_{n+1}-Ax_n)\|^2\\ &\quad +\beta(\|v_{n+1}-v_n\|^2-\|v_{n+1}-v_n-(Ax_{n+1}-Ax_n)\|^2-\|Ax_{n+1}-Ax_n\|^2\\ &= \|x_{n+1}-x_n\|^2+\beta (\beta\|A\|^2-1)\|v_{n+1}-v_n- (Ax_{n+1}-Ax_n)\|^2\\ &\quad +\beta (\|v_{n+1}- v_n\|^2-\|Ax_{n+1}-Ax_n\|^2)\\ &\leq \|x_{n+1}-x_n\|^2+\beta (\|v_{n+1}- v_n\|^2-\|Ax_{n+1}-Ax_n\|^2).\end{aligned} \tag{13}$$

Meanwhile

$$\|v_{n+1}- v_n\| = \|P_Q (I-\alpha_2 g) J_{\mu_2}^{B_2}Ax_{n+1}-P_Q (I-\alpha_2 g) J_{\mu_2}^{B_2}Ax_n\| \leq \|Ax_{n+1}- Ax_n\|,$$

which together with (13), implies that

$$\|z_{n+1}-z_n\| \leq \|x_{n+1} - x_n\|.$$

Next, from (6), we estimate

$$\begin{aligned}\|x_{n+1}- x_n\| &= \|(1-\lambda_n) x_n+\lambda_n y_n-(1-\lambda_{n-1}) x_{n-1}-\lambda_{n-1} y_{n-1}\|\\ &= \|(1-\lambda_n) (x_n-x_{n-1})-(\lambda_n-\lambda_{n-1}) x_{n-1}+\lambda_n(y_n-y_{n-1})+(\lambda_n-\lambda_{n-1}) y_{n-1}\|\\ &\leq (1-\lambda_n)\|x_n-x_{n-1}\| + |\lambda_n-\lambda_{n-1}|\|x_{n-1}\| + \lambda_n\|y_n- y_{n-1}\| + |\lambda_n- \lambda_{n-1}|\|y_{n-1}\|\\ &\leq (1-\lambda_n)\|x_n- x_{n-1}\|+|\lambda_n- \lambda_{n-1}|M_1+ \lambda_n\|y_n- y_{n-1}\|,\end{aligned} \tag{14}$$

where $M_1 = \sup_{n\in \mathrm{N}} \{ \|x_{n-1}\| + \|y_{n-1}\|\}$.
In addition, we observe

$$\begin{aligned}\|y_n-y_{n-1}\| &= \|P_C(I-\alpha_1 f) J_{\mu_1}^{B_1}[\gamma_n u + (1-\gamma_n) z_n]-P_C(I-\alpha_1 f) J_{\mu_1}^{B_1}[\gamma_{n-1}u+(1-\gamma_{n-1}) z_{n-1}]\| \\ &\le \|[\gamma_n u+(1-\gamma_n) z_n]-[\gamma_{n-1}u + (1-\gamma_{n-1}) z_{n-1}]\| \\ &= \|(\gamma_n-\gamma_{n-1}) u+(1-\gamma_n) z_n]-(1-\gamma_{n-1}) z_{n-1}\| \\ &\le |\gamma_n-\gamma_{n-1}|(\|u\| + \|z_{n-1}\|)+ (1-\gamma_n)\|z_n- z_{n-1}\| \\ &\le (1-\gamma_n)\|z_n- z_{n-1}\|+|\gamma_n- \gamma_{n-1}|M_2 \\ &\le (1-\gamma_n)\|x_n- x_{n-1}\|+|\gamma_n- \gamma_{n-1}|M_2, \end{aligned} \tag{15}$$

where $M_2= \sup_{n\in \mathrm{N}} \{ \|u\|+ \|z_{n-1}\|\}$. From (14) and (15), we have

$$\begin{aligned}\|x_{n+1}-x_n\| &\le (1-\lambda_n)\|x_n-x_{n-1}\|+|\lambda_n-\lambda_{n-1}|M_1+\lambda_n[(1-\gamma_n)\|x_n-x_{n-1}\|+ |\gamma_n-\gamma_{n-1}|M_2] \\ &\le (1-\lambda_n\gamma_n)\|x_n-x_{n-1}\|+|\lambda_n- \lambda_{n-1}|M_1+|\gamma_n-\gamma_{n-1}|M_2. \end{aligned} \tag{16}$$

Applying Lemma 2.6 to (16), we deduce that

$$\lim_{n\to\infty}\|x_{n+1}- x_n\| = \lim_{n\to\infty} \lambda_n\|y_n- x_n\| = 0. \tag{17}$$

Take into account of (9) and (12), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|(1-\lambda_n) x_n+\lambda_n y_n-p\|^2 \\ &\le (1-\lambda_n)\|x_n-p\|^2+\lambda_n\|y_n-p\|^2 \\ &= (1-\lambda_n)\|x_n- p\|^2+\lambda_n\|P_C(I- \alpha_1 f) l_n-p\|^2 \\ &\le (1-\lambda_n)\|x_n-p\|^2+\lambda_n\|l_n-p\|^2 \\ &= (1-\lambda_n)\|x_n-p\|^2+\lambda_n\|J_{\mu_1}^{B_1} w_n-p\|^2 \\ &\le (1-\lambda_n)\|x_n-p\|^2+\lambda_n(\|w_n-p\|^2-\|J_{\mu_1}^{B_1} w_n-w_n\|^2) \\ &\le (1-\lambda_n)\|x_n-p\|^2+\lambda_n(\gamma_n\|u-p\|^2+(1-\gamma_n)\|x_n-p\|^2-\|J_{\mu_1}^{B_1} w_n-w_n\|^2), \end{aligned}$$

which implies that

$$\begin{aligned}\lambda_n\|J_{\mu_1}^{B_1} w_n-w_n\|^2 &\le (1-\lambda_n)\|x_n-p\|^2 + \lambda_n(1- \gamma_n)\|x_n-p\|^2-\|x_{n+1}-p\|^2+ \lambda_n\gamma_n\|u-p\|^2 \\ &\le \|x_n- p\|^2- \|x_{n+1}- p\|^2+ \lambda_n\gamma_n\|u- p\|^2 \\ &= (\|x_n- p\|+ \|x_{n+1}- p\|) \|x_{n+1}- x_n\|+ \lambda_n\gamma_n\|u- p\|^2. \end{aligned} \tag{18}$$

Notice that $\lim_{n\to\infty} \gamma_n =0$ and $\{x_n\}$ is bounded. In view of (17) and (18), we deduce

$$\lim_{n\to\infty}\|J_{\mu_1}^{B_1} w_n-w_n\| = \lim_{n\to\infty}\|l_n-w_n\| = 0. \tag{19}$$

Owing to (11) and (12), we have

$$\begin{aligned}\|x_{n+1}- p\|^2 &\le (1- \lambda_n) \|x_n- p\|^2 + \lambda_n\|y_n- p\|^2 \\ &\le (1-\lambda_n)\|x_n-p\|^2 + \lambda_n\|P_C(I-\alpha_1 f) l_n- p\|^2 \end{aligned}$$

$$\begin{aligned}
&= (1-\lambda_n)\|x_n-p\|^2+\lambda_n\|l_n-p\|^2\\
&\le (1-\lambda_n)\|x_n-p\|^2+\lambda_n\|J_{\mu_1}^{B_1}w_n-J_{\mu_1}^{B_1}p\|^2\\
&\le (1-\lambda_n)\|x_n-p\|^2+\lambda_n\|w_n-p\|^2\\
&\le (1-\lambda_n)\|x_n-p\|^2+\lambda_n[\gamma_n\|u-p\|^2+(1-\gamma_n)\|z_n-p\|^2]\\
&\le (1-\lambda_n)\|x_n-p\|^2+\lambda_n\gamma_n\|u-p\|^2\\
&\quad +\lambda_n(1-\gamma_n)[\|x_n-p\|^2-\beta\|J_{\mu_2}^{B_2}Ax_n-Ax_n\|^2-\beta(1-\beta\|A\|^2)\|v_n-Ax_n\|^2]\\
&\le \|x_n-p\|^2+\lambda_n\gamma_n\|u-p\|^2-\lambda_n(1-\gamma_n)[\beta\|J_{\mu_2}^{B_2}Ax_n-Ax_n\|^2\\
&\quad +\beta(1-\beta\|A\|^2)\|v_n-Ax_n\|^2].
\end{aligned}$$

It follows that

$$\begin{aligned}
0 &\le \beta\lambda_n(1-\gamma_n)\|J_{\mu_2}^{B_2}Ax_n-Ax_n\|^2+\beta\lambda_n(1-\gamma_n)(1-\beta\|A\|^2)\|v_n-Ax_n\|^2\\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\lambda_n\gamma_n\|u-p\|^2\\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_{n+1}-x_n\|+\lambda_n\gamma_n\|u-p\|^2\to 0.
\end{aligned}$$

Hence

$$\lim_{n\to\infty}\|v_n-Ax_n\| = \lim_{n\to\infty}\|J_{\mu_2}^{B_2}Ax_n-Ax_n\| = 0, \tag{20}$$

i.e.,

$$\lim_{n\to\infty}\|P_Q(I-\alpha_2 g)J_{\mu_2}^{B_2}Ax_n-Ax_n\| = \lim_{n\to\infty}\|P_Q(I-\alpha_2 g)u_n-Ax_n\| = 0.$$

Notice that $\|z_n-x_n\|\le\beta\|A\|\|Ax_n-v_n\|$. From (20), we obtain that

$$\lim_{n\to\infty}\|z_n-x_n\| = 0. \tag{21}$$

Combining (17), (19) and (20), we have

$$\lim_{n\to\infty}\|P_C(I-\alpha_1 f)x_n-x_n\| = 0,$$

and

$$\|w_n-x_n\|\le\gamma_n\|u-x_n\|+(1-\gamma_n)\|z_n-x_n\|\to 0. \tag{22}$$

Write $x^*=P_\Gamma u$. Next, we prove $\limsup_{n\to\infty}\langle u-x^*, w_n-x^*\rangle\le 0$. Indeed, since $\{w_n\}$ is bounded, there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $\limsup_{n\to\infty}\langle u-x^*, w_n-x^*\rangle = \lim_{i\to\infty}\langle u-x^*, w_{n_i}-x^*\rangle$ and $w_{n_i}\rightharpoonup\hat{p}$, $i\to\infty$. In view of (17), (21) and (22), we have $x_{n_i}\rightharpoonup\hat{p}$, $z_{n_i}\rightharpoonup\hat{p}$, $y_{n_i}\rightharpoonup\hat{p}$, $l_{n_i}\rightharpoonup\hat{p}$, $Ax_{n_i}\rightharpoonup A\hat{p}$ and $u_{n_i}\rightharpoonup\hat{p}$. Moreover, it follows from $x_{n_i}\rightharpoonup\hat{p}$ and $\|P_C(I-\alpha_1 f)x_{n_i}-x_{n_i}\|\to 0$ that $\hat{p}\in VI(C,f)$. Likewise, it follows from $u_{n_i}\rightharpoonup A\hat{p}$ and $\|P_Q(I-\alpha_2 g)u_{n_i}-u_{n_i}\|\to 0$ that $A\hat{p}\in VI(Q,g)$, thus $\hat{p}\in\Omega_1$. On the other hand, by $w_{n_i}\rightharpoonup\hat{p}$, $\lim_{n\to\infty}\|J_{\mu_1}^{B_1}w_n-w_n\| = 0$ and Lemma 2.2, we have $\hat{p}\in \mathrm{Fix}\,(J_{\mu_1}^{B_1})$. Moreover, by $Ax_{n_i}\rightharpoonup A\hat{p}$, $\lim_{n\to\infty}\|J_{\mu_2}^{B_2}Ax_n-Ax_n\| = 0$ and Lemma 2.2, we have $A\hat{p}\in\mathrm{Fix}\,(J_{\mu_2}^{B_2})$, thus, $\hat{p}\in\Omega_2$. Therefore, $\hat{p}\in\Gamma$.

Furthermore, we note that

$$\limsup_{n\to\infty}\langle u-x^*, w_n-x^*\rangle = \langle u-x^*, \hat{p}-x^*\rangle = \langle u-P_\Gamma u, \hat{p}-P_\Gamma u\rangle \leq 0.$$

Finally, we claim that $x_n \to x^*$. Indeed, From (6) and Lemma 2.3, we deduce

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|(1-\lambda_n)\, x_n+\lambda_n\, y_n-x^*\|^2 \\
&\leq (1-\lambda_n)\|x_n-x^*\|^2+\lambda_n\|y_n- x^*\|^2 \\
&\leq (1-\lambda_n)\|x_n-x^*\|^2+\lambda_n\|l_n- x^*\|^2 \\
&\leq (1-\lambda_n)\|x_n-x^*\|^2+\lambda_n\|w_n-x^*\|^2 \\
&\leq (1-\lambda_n)\|x_n-x^*\|^2+\lambda_n\|\gamma_n(u-x^*)+(1-\gamma_n)\,(z_n-x^*)\|^2 \\
&\leq (1-\lambda_n)\|x_n-x^*\|^2+\lambda_n[\, \|(1-\gamma_n)(z_n-x^*)\|^2+2\langle\gamma_n(u-x^*), w_n-x^*\rangle] \\
&\leq (1-\lambda_n)\|x_n-x^*\|^2+\lambda_n(1-\gamma_n)\|z_n- x^*\|^2+2\gamma_n\lambda_n\langle u-x^*, w_n-x^*\rangle \\
&\leq (1-\lambda_n)\|x_n- x^*\|^2+\lambda_n(1- \gamma_n)\|x_n-x^*\|^2+2\gamma_n\, \lambda_n\langle u-x^*, w_n-x^*\rangle \\
&\leq (1-\lambda_n\gamma_n)\, \|x_n-x^*\|^2+2\gamma_n\, \lambda_n\langle u-x^*, w_n-x^*\rangle. \qquad (23)
\end{aligned}$$

Applying Lemma 2.6 to (23), we obtain $\lim_{n\to\infty}\|x_n-x^*\| = 0$. This completes the proof.

Remark 3.1. *Compared with Theorem 3.1 of Zheng et al.* [9], *our Theorem 3.1 extends, improves and develops it in the following aspects:*
(i) Our iterative scheme is more general than it in [9]. *Especially, a Mann iteration scheme is added to construct our iteration process, which is not applied in* [9].
(ii) The result in [9] *can only be applied to solve a split variational inequality problem and a split equilibrium problem, while our result can be applied to solve a split variational inequality problem and a split variational inclusion problem, which makes our result more applicable and valid.*

4. Application

In this section, by applying Theorem 3.1, we prove a strong convergence theorem for approximating the solution of the standard constrained a split convex optimization problem.

Let C be closed convex subset of H. The standard constrained convex optimization problem is to find $x^*\in C$ such that

$$f(x^*) = \min_{x\in C} f(x), \qquad (24)$$

where $f: C\to\mathbf{R}$ is a convex, Fréchet differentiable function. The set of the solutions of (24) is denoted by Φ_f.

Lemma 4.1. ([16]) (Optimality condition) A necessary condition of optimality for a point $x^*\in C$ to be a solution of the minimization problem (4.1) is that x^* solves the variational inequality

$$\langle \nabla f(x^*) , x-x^*\rangle \geq 0, \tag{25}$$

for all $x\in C$. Equivalently, $x^*\in C$ solves the fixed point equation

$$x^* = P_C (I-\lambda\nabla f)\, x^*,$$

for every $\lambda > 0$. If, in addition, f is convex, then the optimality condition (25) is also sufficient.

Theorem 4.1. Let H_i, i=1, 2 be real Hilbert spaces and C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Assume that B_1, B_2: $H_1 \to 2^{H_1}$ are two maximal monotone operators. Let $f : C\to \mathbf{R}$ and $g : Q\to\mathbf{R}$ be real-valued convex functions with the gradients ∇f and ∇g are $\frac{1}{L_f}$ and $\frac{1}{L_g}$ -inverse strongly monotone and continuous with $L_f , L_g > 0$. Let A: $H_1 \to H_2$ be bounded linear operators with adjoint operators A^*. let $\{x_n\}$ be a sequence of H generated by:

Algorithm 2 Let $u\in H_1$ be a fixed point and choose an arbitrary initial point $x_0\in H_1$. For $n\in\mathbf{N}$, let $\{x_n\}$ be a sequence of H_1 generated by:

$$\begin{cases} z_n = x_n - \beta A^*[Ax_n - P_Q(I-\alpha_2\nabla g)J_{\mu_2}^{B_2}Ax_n], \\ y_n = P_C(I-\alpha_1\nabla f)J_{\mu_1}^{B_1}[\gamma_n u + (1-\gamma_n)z_n], \\ x_{n+1} = (1-\lambda_n x_n) + \lambda_n y_n. \end{cases} \tag{26}$$

Assume $\Omega_3 := \{x : x\in\Phi_f, Ax\in\Phi_g\}$, $\Gamma := \Omega_1\cap\Omega_3 \neq \emptyset$, $0 <\alpha_1\leq 2\frac{1}{L_f}$, $0 < \alpha_2\leq 2\frac{1}{L_g}$ and conditions (ii)-(v) hold.

Then the sequence $\{x_n\}$ generated by algorithm (26) converges strongly to the point $p=P_\Gamma u$.

Proof. Since the gradients ∇f and ∇g are $\frac{1}{L_f}$ and $\frac{1}{L_g}$ -inverse strongly monotone, by using Lemma 4.1 and Theorem 3.1, we obtain the desired conclusion directly.

5. Conclusions

We not only extend the iterative algorithm of \cite{9} by adding a Mann iteration scheme to construct our iteration process but also apply our algorithm to a split variational inequality problem and a split variational inclusion problem, which makes our results more applicable and valid. At last, we give an application of the modified algorithm to a split convex optimization problem

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