GLOBAL SMALL SOLUTION TO A CLASS OF CAHN-HILLIARD EQUATION WITH NONLINEAR DIFFUSION

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In this paper, we consider the properties of solutions for the Cauchy problem of a class of Cahn-Hilliard equation with nonlinear diffusion describing the dynamics of phase transitions in ternary oil-water-surfactant systems. We prove the existence of a unique global-in-time solution, analyticity and decay estimates with small initial data in Wiener space $\mathcal{X}_T := \widetilde{L_T^\infty}\mathcal{X}^0 \cap L_T^1\mathcal{X}^2$.

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1. Introduction

In [4, 5], Gompper et al. introduced the free energy functional

$$\mathcal{F}\{u\} = \int \left[f(u) + \frac{1}{2}a(u)|\nabla u|^2 + \frac{\delta}{2}|\Delta u|^2 \right] dx,$$

where $u(x,t)$ implies the local oil-water concentration difference, δ is the second gradient energy coefficient from non-local expansion [3], $a(u)$ is the first gradient coefficient (possibly sign-changing), and $f(u)$ is the free energy density. In the classical Cahn-Hilliard theory, u is conserved, satisfying

$$u_t + \nabla \cdot j = 0,$$

with flux $j = -\nabla \mu$, and chemical potential $\mu = \frac{\delta \mathcal{F}}{\delta u}$. This yields the sixth-order Cahn–Hilliard equation:

$$u_t = \Delta \left[\delta \Delta^2 u - a(u)\Delta u - \tfrac{1}{2}a'(u)|\nabla u|^2 + f'(u) \right]. \tag{1}$$

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Typical choices for $f(u)$ include the Flory-Huggins logarithmic potential or a multi-well potential for three-phase coexistence (oil, water, microemulsion). With oil-water symmetry, $f(u) = (u+1)^2(u^2+h_0)(u-1)^2$ [8, 9]. For increasing amphiphile strength, $a(u)$ becomes negative in the microemulsion phase and is approximated by $a(u) = \kappa_0 + \kappa_1 u^2$ with $\kappa_1 > 0$ [8, 10]. The coefficient $\delta \geq 0$, with $\delta > 0$ giving a sixth-order model. In recent years, the initial-boundary-value problem of equation (1) has attracted considerable attention from mathematicians, particularly regarding well-posedness and asymptotic behavior (see, e.g., Pawlow and Zajaczkowski [8, 10, 9], Schimperna and Pawlow [12, 11], Liu and Wang [7], Zhao [17], and references therein).

The investigation of the the limits as δ tend to zero provides validation of the model as the approximates of the limit fourth order model. Here, we consider equation (1) the fourth-order case $\delta = 0$ in $\mathbb{R}^3$:

$$u_t + \Delta\left[a(u)\Delta u + \tfrac{1}{2}a'(u)|\nabla u|^2 - f'(u)\right] = 0, \tag{2}$$

which corresponds to the Cahn-Hilliard-de Gennes model for binary polymer mixtures [11]. Using f and a as above, the Cauchy problem of equation (2) in $\mathbb{R}^3$ becomes

$$\begin{cases} u_t + \kappa_0\Delta^2 u - 2(1-2h_0)\Delta u \\ \qquad\qquad = -\Delta\left[\kappa_1 u^2\Delta u + \kappa_1 u|\nabla u|^2 - (6u^5 - 8u^3 + 4h_0 u^3)\right], \\ u(x,0) = u_0(x). \end{cases} \tag{3}$$

Recently, Zhao [18] investigated the well-posedness and asymptotic behavior of problem (3) in Sobolev space. By employing higher-order norm estimates of solutions together with a mollifier technique, the author established the local well-posedness of strong solutions. Furthermore, using a pure energy method combined with a standard continuity argument and negative Sobolev norm estimates, the author proved global well-posedness as well as time decay estimates, under the assumption that the H^4-norm of the initial data is sufficiently small. In addition, by means of a parabolic interpolation inequality, a bootstrap argument, and certain weighted estimates, the author also obtained space-time decay estimates for strong solutions.

Since this paper considers problem (3) in the Wiener space, we first briefly introduce it. Lei and Lin [6] were the first to introduce the space

$$\chi_s = \left\{ u \in \mathcal{S}'(\mathbb{R}_+ \times \mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^s \left|\widehat{u}(\xi,t)\right| d\xi < \infty \right\},$$

and they established the global existence of mild solutions for the 3D incompressible Navier–Stokes equations with small initial data in χ_{-1}. Subsequently, Wang and Wang [13] proved the global existence of mild solutions for the MHD equations in χ_{-1}, while Ye [16] extended the study to generalized MHD equations in $\chi_{1-2\alpha}$ with $\alpha \in (\frac{1}{2}, 1)$. In another development, Bae [1] introduced the following time-dependent spaces

$$\begin{cases} \mathcal{X}_{-1} = \left\{ u \in \mathcal{S}'(\mathbb{R}_+ \times \mathbb{R}^3) : \int_{\mathbb{R}^3} \sup_{0 \le t < \infty} \left| |\xi|^{-1} \widehat{u}(\xi, t) \right| d\xi < \infty \right\}, \\ \mathcal{X}_1 = \left\{ u \in \mathcal{S}'(\mathbb{R}_+ \times \mathbb{R}^3) : \int_{\mathbb{R}^3} \int_0^\infty |\xi| \, |\widehat{u}(\xi, t)| \, dt d\xi < \infty \right\}, \end{cases}$$

and used them to reprove Lei and Lin's result. Following Bae's approach [1], Wang and Li [14] investigated the global existence and analyticity of solutions for the 3D generalized Hall-MHD system, and Xiao, Qiu, and Yao [15] studied the 3D regularized MHD equations with fractional Laplacians in the dissipation and diffusion terms. Bae and Lee [2] demonstrated the existence of a unique global-in-time solution, Gevrey regularity, and decay properties for active models with small initial data in scaling-invariant spaces. In contrast to the method of Lei and Lin [6], Bae's method [1] does not allow for a precise characterization of the initial data size in terms of the viscosity coefficient, but it can be used to prove analyticity of the solutions.

We now choose appropriate functional spaces to solve problem (3) under the assumptions $\kappa_0 > 0$ and $h_0 \le \frac{1}{2}$. Define the time-dependent Wiener space

$$\mathcal{X}^s = \left\{ u \in \mathcal{S}'(\mathbb{R}_+ \times R^3) : \int_{\mathbb{R}^3} |\xi|^s |\hat{u}(\xi, t)| \, d\xi < \infty \right\}.$$

If $s = 0$, then $\mathcal{X}^0$ is a Banach algebra with $\mathcal{X}^0 \subset L^\infty$. Next, define the time-dependent function spaces

$$\begin{cases} L^r_T \mathcal{X}_s = \left\{ f \in \mathcal{S}' : \|f\|_{L^r_T \mathcal{X}_s} = \left\| \int_{\mathbb{R}^3} |\xi|^s |\hat{f}(t, \xi)| \, d\xi \right\|_{L^r_T} < \infty \right\}, \\ \widetilde{L^r_T} \mathcal{X}_s = \left\{ f \in \mathcal{S}' : \|f\|_{\widetilde{L^r_T} \mathcal{X}_s} = \int_{\mathbb{R}^3} |\xi|^s \left\| \hat{f}(t, \xi) \right\|_{L^r_T} d\xi < \infty \right\}. \end{cases}$$

In this paper, assuming that the initial data $u_0 \in \mathcal{X}^0$, we consider problem (1.1) in the space

$$\mathcal{X}_T := \widetilde{L^\infty_T} \mathcal{X}^0 \cap L^1_T \mathcal{X}^2.$$

Our first main result on the existence and uniqueness of global-in-time solutions for problem (3) is stated as follows.

Theorem 1.1. *Let $\kappa_0 > 0$, $h_0 \le \frac{1}{2}$. There exists a constant $\varepsilon_0 > 0$ such that for any initial data $u_0 \in \mathcal{X}^0$ with $\|u_0\|_{\mathcal{X}^0} < \varepsilon_0$, there exists a unique global-in-time solution $u \in \mathcal{X}_T$ with $\|u\|_{\mathcal{X}_T} \le C\varepsilon_0$ for all $T > 0$.*

Remark 1.1. *In* [18]*, the author proved that if $u_0 \in H^4(\mathbb{R}^3)$, there exists a global strong solution for problem (3). Moreover, it is not hard to see that if* *the initial data are in the Sobolev space H^s, $s > \frac{3}{2}$, then they are also in the space $\mathcal{X}^0$ (however, this is false for $s = \frac{3}{2}$). Hence, Theorem 1.1 can be seen as an improvement of* [18].

Besides, we establish the following result on the analyticity of solutions for problem (3).

Theorem 1.2. *Let $\kappa_0 > 0$, $h_0 \leq \frac{1}{2}$. Assume that $U(t) = e^{\sqrt{t}\left(\sqrt{\kappa_0}\Lambda^2 + \sqrt{2(1-2h_0)}\Lambda\right)} u$, where $\Lambda = \sqrt{-\Delta}$. Then there exists $\varepsilon_0 > 0$ such that if $u_0 \in \mathcal{X}^0$ with $\|u_0\|_{\mathcal{X}^0} < \varepsilon_0$, there exists a unique solution $U \in \mathcal{X}_T$ with $\|U\|_{\mathcal{X}_T} \lesssim \|u_0\|_{\mathcal{X}^0}$ for all $T > 0$.*

In the end, we show the following theorem on the time decay estimates for problem (3).

Theorem 1.3. *For each $k \in \mathbb{N}$, we have $\|\nabla^k u(t)\|_{\mathcal{X}_0} \leq C_k\left(\sqrt{2(1-2h_0)\, t}\right)^{-k}$, where C_k is a positive constant depending on k and ε_0.*

The paper is organized as follows. In Section 2, we present the proof of Theorem 1.1, which concerns the global well-posedness of solutions. Section 3 is dedicated to demonstrating the analyticity of solutions, as stated in Theorem 1.2. In Section 4, we prove Corollary 1.3, also regarding solution analyticity. Finally, we adopt the notation $A \lesssim B$ to mean $A \leq cB$, and $A \simeq B$ to mean $A = cB$, where $c > 0$ denotes an absolute constant.

2. **Proof of Theorem 1.1**

To find solutions in the space $\mathcal{X}_T$, we rewrite equation $(3)_1$ in its integral formulation

$$u(t) = e^{t(-\kappa_0\Delta^2 + 2(1-2h_0)\Delta)} u_0$$
$$-\int_0^t e^{(t-s)(-\kappa_0\Delta^2+2(1-2h_0)\Delta)} \Delta\left[\kappa_1 u^2 \Delta u + \kappa_1 u |\nabla u|^2 - (6u^5 - 8u^3 + 4h_0 u^3)\right](s)\, ds. \tag{4}$$

Taking the Fourier transform of (4), we obtain

$$\begin{aligned}
\hat{u}(t,\xi) =& e^{t(-\kappa_0|\xi|^4 - 2(1-2h_0)|\xi|^2)} \hat{u}_0(\xi) \\
&+ \kappa_1 \int_0^t e^{(t-s)(-\kappa_0|\xi|^4 - 2(1-2h_0)|\xi|^2)} |\xi|^2 \widehat{u^2\Delta u}(s,\xi) ds \\
&+ \kappa_1 \int_0^t e^{(t-s)(-\kappa_0|\xi|^4 - 2(1-2h_0)|\xi|^2)} |\xi|^2 \widehat{u|\nabla u|^2}(s,\xi) ds \\
&- \int_0^t e^{(t-s)(-\kappa_0|\xi|^4 - 2(1-2h_0)|\xi|^2)} |\xi|^2 (6u^5 - \widehat{8u^3 +} 4h_0 u^3)(s,\xi) ds \\
=& I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{5}$$

We will bound I_1-I_4 in the following one by one. Note that

$$\int_0^\infty |\xi|^2 e^{t\left(-\kappa_0|\xi|^4-2(1-2h_0)|\xi|^2\right)} dt \le C, \quad \int_0^\infty |\xi|^4 e^{t\left(-\kappa_0|\xi|^4-2(1-2h_0)|\xi|^2\right)} dt \le C. \tag{6}$$

Hence, I_1 satisfies

$$\|I_1\|_{\mathcal{X}_T} = \left\| e^{t(-\kappa_0|\xi|^4-2(1-2h_0)|\xi|^2)} \hat{u_0}(\xi) \right\|_{\mathcal{X}_T} \lesssim \|u_0\|_{\mathcal{X}_0}. \tag{7}$$

Next, using (6), taking L_T^∞ norm to I_2, it yields that

$$\begin{aligned}
\sup_{0<t<T} |I_2| \lesssim& \int_0^T \left(\int_{\mathbb{R}^3} |\widehat{\Delta u}(s,\xi-\eta)||\widehat{u}(s,\eta-\zeta)||\widehat{u}(s,\zeta)|d\zeta \right. \\
&+ \int_{\mathbb{R}^3} |\widehat{u}(s,\xi-\eta)||\widehat{\Delta u}(s,\eta-\zeta)||\widehat{u}(s,\zeta)|d\zeta \\
&\left. + \int_{\mathbb{R}^3} |\widehat{\Delta u}(s,\xi-\eta)||\widehat{u}(s,\eta-\zeta)||\widehat{\Delta u}(s,\zeta)|d\zeta \right) dsds \\
\lesssim& \int_0^T \left(\int_{\mathbb{R}^3} (|\xi-\eta|^2|\eta-\zeta|^0|\zeta|^0 + |\xi-\eta|^0|\eta-\zeta|^2|\zeta|^0 \right. \\
&+|\xi-\eta|^0|\eta-\zeta|^0|\zeta|^2) \\
&\cdot (|\widehat{u}(\xi-\eta,s)||\widehat{u}(\eta-\zeta,s)||\widehat{u}(\zeta,s)|d\zeta)\, ds.
\end{aligned} \tag{8}$$

Integrating (8) in ξ, we deduce that

$$\|I_2\|_{\widetilde{L_T^\infty}\mathcal{X}_0} \lesssim \|u\|^2_{\widetilde{L_T^\infty}\mathcal{X}_0} \|u\|_{L_T^1\mathcal{X}_2}. \tag{9}$$

Similar to (8), by using (6), we have

$$\begin{aligned}
|\xi|^2\|I_2\|_{L_T^1} \lesssim& \int_0^T \left(\int_{\mathbb{R}^3} |\widehat{\Delta u}(s,\xi-\eta)||\widehat{u}(s,\eta-\zeta)||\widehat{u}(s,\zeta)|d\zeta \right. \\
&+ \int_{\mathbb{R}^3} |\widehat{u}(s,\xi-\eta)||\widehat{\Delta u}(s,\eta-\zeta)||\widehat{u}(s,\zeta)|d\zeta \\
&\left. + \int_{\mathbb{R}^3} |\widehat{\Delta u}(s,\xi-\eta)||\widehat{u}(s,\eta-\zeta)||\widehat{\Delta u}(s,\zeta)|d\zeta \right) ds \\
\lesssim& \int_0^T \left(\int_{\mathbb{R}^3} (|\xi-\eta|^2|\eta-\zeta|^0|\zeta|^0 + |\xi-\eta|^0|\eta-\zeta|^2|\zeta|^0 \right. \\
&+|\xi-\eta|^0|\eta-\zeta|^0|\zeta|^2) \\
&\cdot (|\widehat{u}(\xi-\eta,s)||\widehat{u}(\eta-\zeta,s)||\widehat{u}(\zeta,s)|d\zeta)\, ds.
\end{aligned}$$

which implies that

$$\|I_2\|_{L_T^1\mathcal{X}_2} \lesssim \|u\|^2_{\widetilde{L_T^\infty}\mathcal{X}_0} \|u\|_{L_T^1\mathcal{X}_2}. \tag{10}$$

Combining (9) and (10) together gives

$$\|I_2\|_{\mathcal{X}_T} \lesssim \|u\|^2_{\widetilde{L^\infty_T}\mathcal{X}_0}\|u\|_{L^1_T\mathcal{X}_2} \lesssim \|u\|^3_{\mathcal{X}_T}. \tag{11}$$

The term I_3 will be considered in the follwing . By using (6) , taking L^∞_T norm to I_3 , we obtain

$$\begin{aligned}
\sup_{0<t<T}|I_3| \lesssim & \int_0^T \bigg(\int_{\mathbb{R}^3} |\widehat{u}(s,\xi-\eta)||\widehat{\nabla u}(s,\eta-\zeta)||\widehat{\nabla u}(s,\zeta)|d\zeta \\
& + \int_{\mathbb{R}^3} |\widehat{\nabla u}(s,\xi-\eta)||\widehat{u}(s,\eta-\zeta)||\widehat{\nabla u}(s,\zeta)|d\zeta \\
& + \int_{\mathbb{R}^3} |\widehat{\nabla u}(s,\xi-\eta)||\widehat{\nabla u}(s,\eta-\zeta)||\widehat{u}(s,\zeta)|d\zeta \bigg) dsds \\
\lesssim & \int_0^T \bigg(\int_{\mathbb{R}^3} (|\xi-\eta|^0|\eta-\zeta||\zeta| + |\xi-\eta||\eta-\zeta|^0|\zeta| \\
& + |\xi-\eta||\eta-\zeta||\zeta|^0) \\
& \cdot (|\widehat{u}(\xi-\eta,s)||\widehat{u}(\eta-\zeta,s)||\widehat{u}(\zeta,s)|d\zeta)\, ds \\
\lesssim & \int_0^T \bigg(\int_{\mathbb{R}^3} (|\xi-\eta|^2|\eta-\zeta|^0|\zeta|^0 + |\xi-\eta|^0|\eta-\zeta|^2|\zeta|^0 \\
& + |\xi-\eta|^0|\eta-\zeta|^0|\zeta|^2) \\
& \cdot (|\widehat{u}(\xi-\eta,s)||\widehat{u}(\eta-\zeta,s)||\widehat{u}(\zeta,s)|d\zeta)\, ds,
\end{aligned} \tag{12}$$

where we have used the classical inequality $ab+bc+ac \lesssim a^2+b^2+c^2$ ($\forall a,b,c \in \mathbb{R}$) in the above inequality. Integrating (12) in ξ, we deduce that

$$\|I_3\|_{\widetilde{L^\infty_T}\mathcal{X}_0} \lesssim \|u\|^2_{\widetilde{L^\infty_T}\mathcal{X}_0}\|u\|_{L^1_T\mathcal{X}_2}. \tag{13}$$

Similar to (12), by using (6), we have

$$\begin{aligned}
|\xi|^2\|I_3\|_{L^1_T} \lesssim & \int_0^T \bigg(\int_{\mathbb{R}^3} |\widehat{u}(s,\xi-\eta)||\widehat{\nabla u}(s,\eta-\zeta)||\widehat{\nabla u}(s,\zeta)|d\zeta \\
& + \int_{\mathbb{R}^3} |\widehat{\nabla u}(s,\xi-\eta)||\widehat{u}(s,\eta-\zeta)||\widehat{\nabla u}(s,\zeta)|d\zeta \\
& + \int_{\mathbb{R}^3} |\widehat{\nabla u}(s,\xi-\eta)||\widehat{\nabla u}(s,\eta-\zeta)||\widehat{u}(s,\zeta)|d\zeta \bigg) dsds \\
& \cdot (|\widehat{u}(\xi-\eta,s)||\widehat{u}(\eta-\zeta,s)||\widehat{u}(\zeta,s)|d\zeta)\, ds,
\end{aligned}$$

which implies that

$$\|I_3\|_{L^1_T\mathcal{X}_2} \lesssim \|u\|^2_{\widetilde{L^\infty_T}\mathcal{X}_0}\|u\|_{L^1_T\mathcal{X}_2}. \tag{14}$$

Combining (13) and (14) together gives

$$\|I_3\|_{\mathcal{X}_T} \lesssim \|u\|^2_{\widetilde{L^\infty_T}\mathcal{X}_0}\|u\|_{L^1_T\mathcal{X}_2} \lesssim \|u\|^3_{\mathcal{X}_T}. \tag{15}$$

Besides, by using (6), taking L_T^∞ to I_4, we obtain

$$\begin{aligned}
\sup_{0<t<T}|I_4| \lesssim & \int_0^t |\xi|^2 \int_{\mathbb{R}^3} (\widehat{u^5}(s,\xi) + \widehat{u^3}(s,\xi)) d\xi ds \\
\lesssim & \int_0^t \int_{\mathbb{R}^3} (|\xi-\eta|^2 |\eta-\zeta|^0 |\zeta-\varsigma|^0 |\varsigma-\upsilon|^0 |\upsilon|^0 \\
& + |\xi-\eta|^0 |\eta-\zeta|^2 |\zeta-\varsigma|^0 |\varsigma-\upsilon|^0 |\upsilon|^0 \\
& + |\xi-\eta|^0 |\eta-\zeta|^0 |\zeta-\varsigma|^2 |\varsigma-\upsilon|^0 |\upsilon|^0 \\
& + |\xi-\eta|^0 |\eta-\zeta|^0 |\zeta-\varsigma|^0 |\varsigma-\upsilon|^2 |\upsilon|^0 \\
& + |\xi-\eta|^0 |\eta-\zeta|^0 |\zeta-\varsigma|^0 |\varsigma-\upsilon|^0 |\upsilon|^2) \\
& \cdot (|\widehat{u}(s,\xi-\eta)||\widehat{u}(s,\eta-\zeta)||\widehat{u}(s,\zeta-\varsigma||\widehat{u}(s,\varsigma-\upsilon)||\widehat{u}(s,\upsilon)| d\upsilon ds \\
& + \int_0^t \int_{\mathbb{R}^3} (|\xi-\eta|^2 |\eta-\zeta|^0 |\zeta|^0 + |\xi-\eta|^0 |\eta-\zeta|^2 |\zeta|^0 \\
& + |\xi-\eta|^0 |\eta-\zeta|^0 |\zeta|^2) \\
& \cdot (|\widehat{u}(s,\xi-\eta)||\widehat{u}(s,\eta-\zeta)||\widehat{u}(s,\zeta)| d\zeta ds.
\end{aligned} \tag{16}$$

Integrating (16) in ξ, we deduce that

$$\|I_4\|_{\widetilde{L_T^\infty}\mathcal{X}_0} \lesssim \|u\|^4_{\widetilde{L_T^\infty}\mathcal{X}_0} \|u\|_{L_T^1\mathcal{X}_2} + \|u\|^2_{\widetilde{L_T^\infty}\mathcal{X}_0} \|u\|_{L_T^1\mathcal{X}_2}. \tag{17}$$

Moreover, by using (6), we derive that

$$\begin{aligned}
|\xi|^2 \|I_4\|_{L_T^1} \lesssim & \int_0^t |\xi|^2 \int_{\mathbb{R}^3} (\widehat{u^5}(s,\xi) + \widehat{u^4}(s,\xi) + \widehat{u^3}(s,\xi) + \widehat{u^2}(s,\xi)) d\xi ds \\
\lesssim & \int_0^t \int_{\mathbb{R}^3} (|\xi-\eta|^2 |\eta-\zeta|^0 |\zeta-\varsigma|^0 |\varsigma-\upsilon|^0 |\upsilon|^0 \\
& + |\xi-\eta|^0 |\eta-\zeta|^2 |\zeta-\varsigma|^0 |\varsigma-\upsilon|^0 |\upsilon|^0 \\
& + |\xi-\eta|^0 |\eta-\zeta|^0 |\zeta-\varsigma|^2 |\varsigma-\upsilon|^0 |\upsilon|^0 \\
& + |\xi-\eta|^0 |\eta-\zeta|^0 |\zeta-\varsigma|^0 |\varsigma-\upsilon|^2 |\upsilon|^0 \\
& + |\xi-\eta|^0 |\eta-\zeta|^0 |\zeta-\varsigma|^0 |\varsigma-\upsilon|^0 |\upsilon|^2) \\
& \cdot (|\widehat{u}(s,\xi-\eta)||\widehat{u}(s,\eta-\zeta)||\widehat{u}(s,\zeta-\varsigma||\widehat{u}(s,\varsigma-\upsilon)||\widehat{u}(s,\upsilon)| d\upsilon ds \\
& + \int_0^t \int_{\mathbb{R}^3} (|\xi-\eta|^2 |\eta-\zeta|^0 |\zeta|^0 + |\xi-\eta|^0 |\eta-\zeta|^2 |\zeta|^0 \\
& + |\xi-\eta|^0 |\eta-\zeta|^0 |\zeta|^2) \\
& \cdot (|\widehat{u}(s,\xi-\eta)||\widehat{u}(s,\eta-\zeta)||\widehat{u}(s,\zeta)| d\zeta ds.
\end{aligned}$$

which implies that

$$\|I_4\|_{L_T^1\mathcal{X}_2} \lesssim \|u\|^4_{\widetilde{L_T^\infty}\mathcal{X}_0} \|u\|_{L_T^1\mathcal{X}_2} + \|u\|^2_{\widetilde{L_T^\infty}\mathcal{X}_0} \|u\|_{L_T^1\mathcal{X}_2}. \tag{18}$$

Combining (17) and (18) together gives

$$\begin{aligned}\|I_4\|_{\mathfrak{X}_T} \lesssim&\|u\|^4_{\widetilde{L^\infty_T}\mathfrak{X}_0}\|u\|_{L^1_T\mathfrak{X}_2}+\|u\|^2_{\widetilde{L^\infty_T}\mathfrak{X}_0}\|u\|_{L^1_T\mathfrak{X}_2}\\ \lesssim&\|u\|^5_{\mathfrak{X}_T}+\|u\|^3_{\mathfrak{X}_T}.\end{aligned}$$

Combining (5), (7), (11) and (15) together, by using Young's inequality, we derive that

$$\|u\|_{\mathfrak{X}_T} \lesssim\|u_0\|_{\mathfrak{X}_0}+\|u\|^3_{\mathfrak{X}_T}+\|u\|^5_{\mathfrak{X}_T}\le\|u_0\|_{\mathfrak{X}_0}+\frac{1}{2}\|u\|_{\mathfrak{X}_T}+C\|u\|^5_{\mathfrak{X}_T},$$

which implies that

$$\|u\|_{\mathfrak{X}_T} \lesssim\|u_0\|_{\mathfrak{X}_0}+\|u\|^5_{\mathfrak{X}_T}.$$

Hence , if $\|u_0\|$is sufficiently small , we can obtain

$$\|u\|_{\mathfrak{X}_T}\lesssim\|u_0\|_{\mathfrak{X}_0}\lesssim\varepsilon_0,$$

which complete the proof of the existence part of Theorem 1.1.

In the following, we prove the uniqueness part. Assume that there exist two solutions u_1 and u_2. Let $u=u_1-u_2$, then it satisfies

$$\begin{aligned}&u_t+\kappa_0\Delta^2u-2(1-2h_0)\Delta u\\ =&-\Delta\left[\kappa_1u_1^2\Delta u+\kappa_1u(u_1+u_2)\Delta u_2+\kappa_1u|\nabla u_1|^2+\kappa_1u_2(\nabla u_1+\nabla u_2)\cdot\nabla u\right.\\ &\left.-6u(u_1^4+u_1^3u_2+u_1^2u_2^2+u_1u_2^3+u_2^4)+(4h_0-8)u(u_1^2+u_1u_2+u_2^2)\right].\end{aligned}\tag{19}$$

It then follows from (19) that

$$\begin{aligned}\|u\|_{\mathfrak{X}_T}\lesssim&\|u\|_{\mathfrak{X}_T}\big(\|u_1\|^2_{\mathfrak{X}_T}+\|u_2\|^2_{\mathfrak{X}_T}+\|u_1\|^4_{\mathfrak{X}_T}+\|u_1\|^3_{\mathfrak{X}_T}\|u_2\|_{\mathfrak{X}_T}\\ &+\|u_1\|^2_{\mathfrak{X}_T}\|u_2\|^2_{\mathfrak{X}_T}+\|u_1\|_{\mathfrak{X}_T}\|u_2\|^3_{\mathfrak{X}_T}+\|u_2\|^4_{\mathfrak{X}_T}\big)\\ \lesssim&\big(\varepsilon_0^2+\varepsilon_0^4\big)\|u\|_{\mathfrak{X}_T}.\end{aligned}\tag{20}$$

By restricting ε_0 in (20) further to satisfying $C(\varepsilon_0^2+\varepsilon_0^4)<1$, we complete the proof of uniqueness.

3. **Proof of Theorem 1.2**

Assume that $\hat{U}(t,\xi)=e^{\sqrt{t}\left(\sqrt{\kappa_0}|\xi|^2+\sqrt{2(1-2h_0)}|\xi|\right)}\hat{u}(t,\xi)$, we easily obtain which satisfies

$$\begin{aligned}\left|\hat{U}(t,\xi)\right|\le& e^{\sqrt{t}\left(\sqrt{\kappa_0}|\xi|^2+\sqrt{2(1-2h_0)}|\xi|\right)-t\left(\kappa_0|\xi|^4+2(1-2h_0)|\xi|^2\right)}|\hat{u}_0(t,\xi)|\\ &+\int_0^t e^{\sqrt{t}\left(\sqrt{\kappa_0}|\xi|^2+\sqrt{2(1-2h_0)}|\xi|\right)-(t-s)\left(\kappa_0|\xi|^4+2(1-2h_0)|\xi|^2\right)}|\xi|^2\\ &\cdot\int_{\mathbb{R}^3}\left[|\kappa_1\widehat{u^2\Delta u}|+|\kappa_1\widehat{u|\nabla u|^2}|+\left|6\hat{u}^5-8\hat{u}^3+4h_0\hat{u}^3\right|\right](s)\,d\xi\,ds.\end{aligned}\tag{21}$$

Since the expression $e^{\sqrt{t}\left(\sqrt{\kappa_0}|\xi|^2+\sqrt{2(1-2h_0)}|\xi|\right)-\frac{t}{2}\left(\kappa_0|\xi|^4+2(1-2h_0)|\xi|^2\right)}$ is uniformly bounded in time, the linear part behaves analogously to the linear part in equation (5). Consequently, from this point onward, we focus solely on the nonlinear terms. Note that

$$
\begin{aligned}
&\int_0^t e^{\sqrt{t}\left(\sqrt{\kappa_0}|\xi|^2+\sqrt{2(1-2h_0)}|\xi|\right)-(t-s)\left(\kappa_0|\xi|^4+2(1-2h_0)|\xi|^2\right)} \int_{\mathbb{R}^3} |\xi|^2\kappa_1\widehat{u^2\Delta u}(s,\xi)\, d\xi\, ds \\
\lesssim &\int_0^t e^{(\sqrt{t}-\sqrt{s})\left(\sqrt{\kappa_0}|\xi|^2+\sqrt{2(1-2h_0)}|\xi|\right)-\frac{t-s}{2}\left(\kappa_0|\xi|^4+2(1-2h_0)|\xi|^2\right)} e^{-\frac{t-s}{2}\left(\kappa_0|\xi|^4+2(1-2h_0)|\xi|^2\right)} \\
&\cdot e^{\sqrt{s}\left(\sqrt{\kappa_0}|\xi|^2+\sqrt{2(1-2h_0)}|\xi|\right)} \int_{\mathbb{R}^3} |\xi|^2|\hat{u}(\xi-\eta,s)||\hat{u}(\eta-\zeta,s)||\hat{u}(\zeta,s)|d\zeta ds \\
\lesssim &\int_0^t e^{(\sqrt{t}-\sqrt{s})\left(\sqrt{\kappa_0}|\xi|^2+\sqrt{2(1-2h_0)}|\xi|\right)-\frac{t-s}{2}\left(\kappa_0|\xi|^4+2(1-2h_0)|\xi|^2\right)} e^{-\frac{t-s}{2}\left(\kappa_0|\xi|^4+2(1-2h_0)|\xi|^2\right)} \\
&\cdot e^{\sqrt{s}\left(\sqrt{\kappa_0}\left(|\xi|^2-|\xi-\eta|^2-|\eta-\zeta|^2-|\zeta|^2\right)+\sqrt{2(1-2h_0)}(|\xi|-|\xi-\eta|-|\eta-\zeta|-|\zeta|\right)} \\
&\cdot \int_{\mathbb{R}^3} |\xi|^2|\hat{U}(\xi-\eta,s)||\hat{U}(\eta-\zeta,s)||\hat{U}(\zeta,s)|d\zeta ds.
\end{aligned}
\tag{22}
$$

Simple calculations show that

$$
e^{(\sqrt{t}-\sqrt{s})\left(\sqrt{\kappa_0}|\xi|^2+\sqrt{2(1-2h_0)}|\xi|\right)-\frac{t-s}{2}\left(\kappa_0|\xi|^4+2(1-2h_0)|\xi|^2\right)} \lesssim 1,
$$

and

$$
e^{\sqrt{s}\left(\sqrt{\kappa_0}\left(|\xi|^2-|\xi-\eta|^2-|\eta-\zeta|^2-|\zeta|^2\right)+\sqrt{2(1-2h_0)}(|\xi|-|\xi-\eta|-|\eta-\zeta|-|\zeta|)\right)} \lesssim 1.
$$

Therefore, we can estimate (22) as

$$
\begin{aligned}
&\int_0^t e^{\sqrt{t}\left(\sqrt{\kappa_0}|\xi|^2+\sqrt{2(1-2h_0)}|\xi|\right)-(t-s)\left(\kappa_0|\xi|^4+2(1-2h_0)|\xi|^2\right)} \int_{\mathbb{R}^3} |\xi|^2\kappa_1\widehat{u^2\Delta u}(s,\xi)\, d\xi\, ds \\
\lesssim &\int_0^t e^{-\frac{t-s}{2}\left(\kappa_0|\xi|^4+2(1-2h_0)|\xi|^2\right)} \int_{\mathbb{R}^3} |\xi|^2[|\Delta\hat{U}(\xi-\eta,s)||\hat{U}(\eta-\zeta,s)||\hat{U}(\zeta,s)| \\
&+|\hat{U}(\xi-\eta,s)||\Delta\hat{U}(\eta-\zeta,s)||\hat{U}(\zeta,s)| \\
&+|\hat{U}(\xi-\eta,s)||\hat{U}(\eta-\zeta,s)||\Delta\hat{U}(\zeta,s)|]d\zeta ds.
\end{aligned}
$$

The remaining terms on the right-hand side of (21) can be estimated using the same approach as above. Hence, we easily deduce that

$$
\begin{aligned}
&|\hat{U}(t,\xi)| \lesssim e^{-\frac{t}{2}\left(\kappa_0|\xi|^4+2(1-h_0)|\xi|^2\right)}|\hat{u}_0(t,\xi)| \\
&+\int_0^t e^{-\frac{t-s}{2}\left(\kappa_0|\xi|^4+2(1-2h_0)|\xi|^2\right)}\int_{\mathbb{R}^3}|\xi|^2[|\hat{\Delta U}(\xi-\eta,s)||\hat{U}(\eta-\zeta,s)||\hat{U}(\zeta,s)| \\
&+|\hat{U}(\xi-\eta,s)||\hat{\Delta U}(\eta-\zeta,s)||\hat{U}(\zeta,s)| \\
&+|\hat{U}(\xi-\eta,s)||\hat{U}(\eta-\zeta,s)||\hat{\Delta U}(\zeta,s)|]d\zeta ds \\
&+\int_0^t e^{-\frac{t-s}{2}\left(\kappa_0|\xi|^4+2(1-2h_0)|\xi|^2\right)}\int_{\mathbb{R}^3}|\xi|^2\left(|\hat{U}(\xi-\eta,s)||\hat{U}(\eta-\zeta,s)||\hat{U}(\zeta,s)|\right)d\zeta\,ds \\
&+\int_0^t e^{-\frac{t-s}{2}\left(\kappa_0|\xi|^4+2(1-2h_0)|\xi|^2\right)}\int_{\mathbb{R}^3}|\xi|^2[|\hat{\nabla U}(\xi-\eta,s)||\hat{\nabla U}(\eta-\zeta,s)||\hat{U}(\zeta,s)| \\
&+|\hat{\nabla U}(\xi-\eta,s)||\hat{U}(\eta-\zeta,s)||\hat{\nabla U}(\zeta,s)| \\
&+|\hat{U}(\xi-\eta,s)||\hat{\nabla U}(\eta-\zeta,s)||\hat{\nabla U}(\zeta,s)|]d\zeta ds \\
&+\int_0^t e^{-\frac{t-s}{2}\left(\kappa_0|\xi|^4+2(1-2h_0)|\xi|^2\right)}\int_{\mathbb{R}^3}|\xi|^2\left(|\hat{U}(\xi-\eta,s)||\hat{U}(\eta-\zeta,s)||\hat{U}(\zeta,s)|\right)d\zeta\,ds \\
&+\int_0^t e^{-\frac{t-s}{2}\left(\kappa_0|\xi|^4+2(1-2h_0)|\xi|^2\right)} \\
&\cdot\int_{\mathbb{R}^3}|\xi|^2\left(|\hat{U}(\xi-\eta,s)|\,|\hat{U}(\eta-\zeta,s)|\,|\hat{U}(\zeta-\sigma,s)|\,|\hat{U}(\sigma-\gamma,s)|\,|\hat{U}(\gamma,s)|\right)d\gamma\,ds \\
&+\int_0^t e^{-\frac{t-s}{2}\left(\kappa_0|\xi|^4+2(1-2h_0)|\xi|^2\right)}\int_{\mathbb{R}^3}|\xi|^2\left(|\hat{U}(\xi-\eta,s)||\hat{U}(\eta-\zeta,s)||\hat{U}(\zeta,s)|\right)d\zeta\,ds.
\end{aligned}
$$

Following the proof of Theorem 1.1 step by step then yields $\|U\|_{\mathcal{X}^T} \lesssim \|u_0\|_{\mathcal{X}^0}$. This completes the proof.

4. **Proof of Theorem 1.3**

For each $k \in \mathbb{N}$, we write

$$
\begin{aligned}
\left\|\nabla^k u(t)\right\|_{\mathcal{X}_0} &= \int_{\mathbb{R}^N}|\xi|^k\,|\hat{u}\,(t,\xi)|\;d\xi \\
&= \int_{\mathbb{R}^N}|\xi|^k e^{-\sqrt{t}\left(\sqrt{\kappa_0}|\xi|^2+\sqrt{2(1-2h_0)}|\xi|\right)}\left|\hat{U}\,(t,\xi)\right|\;d\xi.
\end{aligned}
$$

Simple calculation shows that

$$
\begin{aligned}
|\xi|^k e^{-\sqrt{t}\left(\sqrt{\kappa_0}|\xi|^2+\sqrt{2(1-2h_0)}|\xi|\right)} &= \left(\sqrt{2(1-2h_0)t}\right)^{-k}\left(\sqrt{2(1-2h_0)t}|\xi|\right)^k \\
&\quad \cdot e^{-\sqrt{t}\left(\sqrt{\kappa_0}|\xi|^2+\sqrt{2(1-2h_0)}|\xi|\right)} \\
&\le D_k\left(\sqrt{2(1-2h_0)t}\right)^{-k},
\end{aligned}
$$

where D_k is a positive constant depends only on k and ε_0. Using the estimate from Theorem 1.2 together with the embedding $\widetilde{L}_T^\infty\mathcal{X}_0 \subset L_T^\infty\mathcal{X}_0$, we derive that

$$
\left\|\nabla^k u(t)\right\|_{\mathcal{X}_0} \le D_k\left(\sqrt{2(1-2h_0)t}\right)^{-k}\|U(t)\|_{\mathcal{X}_0} \le C_k\left(\sqrt{2(1-2h_0)t}\right)^{-k}.
$$

where C_k is a positive constant depending on k and ε_0, this complete the proof of Theorem 1.3.

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