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AN EKELAND-TYPE VARIATIONAL PRINCIPLE IN STRICTLY DIAGONAL PARTIAL METRIC SPACES

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We establish an Ekeland-type variational principle in a class of partial metric spaces whose diagonal map is strictly Lipschitz with respect to the associated symmetrized metric. This assumption gives a two-sided comparison between the induced quasi-metric and the symmetrized metric, and it permits the transfer of variational arguments from the complete metric framework to the partial metric setting. As consequences, we obtain a penalized strict minimization principle, Caristi-type fixed point theorems, a Takahashi-type minimization result, and common fixed point results for semigroups and commuting resolvent-type families. We also give a set-valued variational principle on closed graphs, an asymmetric equilibrium principle, and a discrete variational generation scheme for nonlinear evolution models. Explicit examples illustrate the abstract results.

Keywords: variational principle, partial metric, strict diagonal condition, Caristi theorem, semigroup, resolvent family

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1. **Introduction**

Variational principles are central tools in nonlinear analysis, optimization, and fixed point theory. Ekeland's variational principle is especially useful because it produces approximate minimizers with a strong stability property [2]. It also implies minimization theorems, descent principles, and fixed point results of Caristi type [1], as well as minimization results in the spirit of Takahashi [6].

Partial metric spaces, introduced by Matthews [3], extend ordinary metric spaces by allowing a point to have a nonzero self-distance. This feature is natural in several generalized topological models, but it also creates a technical difficulty: the topology is usually encoded by an asymmetric quasi-metric, while many variational arguments require a complete metric framework. The aim of the present paper is to show that this difficulty can be removed in a useful quantitative class of partial metrics.

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Let ϱ be a partial metric on a set X. We associate to ϱ the diagonal map

$$\delta_\varrho(x) = \varrho(x, x),$$

the quasi-metric

$$\eta_\varrho(x, y) = \varrho(x, y) - \varrho(x, x),$$

and the symmetrized metric

$$\sigma_\varrho(x, y) = 2\varrho(x, y) - \varrho(x, x) - \varrho(y, y).$$

We focus on the strictly diagonal regime, namely the existence of $\theta \in [0, 1)$ such that

$$|\delta_\varrho(x) - \delta_\varrho(y)| \le \theta\, \sigma_\varrho(x, y), \qquad x, y \in X.$$

Under this assumption, η_ϱ and σ_ϱ are quantitatively equivalent and the partial metric topology coincides with the metric topology induced by σ_ϱ. Hence complete variational statements in the metric model (X, σ_ϱ) can be translated back to the partial metric framework.

The set-valued and asymmetric parts of the paper are connected with the equilibrium-problem framework of Blum and Oettli [4]. The discrete generation scheme is motivated by nonlinear semigroup generation methods in the sense of Crandall and Liggett [5].

The paper is organized as follows. Section 2 recalls the basic material and proves the metric comparison. Section 3 gives the Ekeland-type variational principle and its quasi-metric form. Section 4 derives fixed point and minimization consequences. Section 5 presents concrete models. Section 6 extends the method to set-valued, asymmetric, and evolutionary schemes. Section 7 concludes the paper.

2. **Preliminaries**

Definition 2.1. *A mapping $\varrho : X \times X \to [0, \infty)$ is called a partial metric on a nonempty set X if, for all $x, y, z \in X$,*
(pm1) $x = y$ if and only if $\varrho(x, x) = \varrho(x, y) = \varrho(y, y)$;
(pm2) $\varrho(x, x) \le \varrho(x, y)$;
(pm3) $\varrho(x, y) = \varrho(y, x)$;
(pm4) $\varrho(x, z) \le \varrho(x, y) + \varrho(y, z) - \varrho(y, y)$.
The pair (X, ϱ) is then called a partial metric space.

For a partial metric ϱ, define

$$\delta_\varrho(x) = \varrho(x, x), \qquad \eta_\varrho(x, y) = \varrho(x, y) - \varrho(x, x)$$

and

$$\sigma_\varrho(x, y) = 2\varrho(x, y) - \varrho(x, x) - \varrho(y, y).$$

It is classical in partial metric theory that η_ϱ is a quasi-metric and that σ_ϱ is a genuine metric [3].

Definition 2.2. *The partial metric ϱ is said to be strictly diagonal if there exists $\theta \in [0,1)$ such that*

$$|\delta_\varrho(x) - \delta_\varrho(y)| \le \theta\, \sigma_\varrho(x,y), \qquad x, y \in X.$$

The following identity follows immediately from the definitions:

$$\eta_\varrho(x,y) = \frac{1}{2}\Big(\sigma_\varrho(x,y) - \delta_\varrho(x) + \delta_\varrho(y)\Big).$$

This identity gives the comparison between the quasi-metric and the symmetrized metric.

Proposition 2.1. *Let (X,ϱ) be a strictly diagonal partial metric space with constant $\theta \in [0,1)$. Then, for all $x, y \in X$,*

$$\frac{1-\theta}{2}\,\sigma_\varrho(x,y) \le \eta_\varrho(x,y) \le \frac{1+\theta}{2}\,\sigma_\varrho(x,y).$$

In particular, η_ϱ and σ_ϱ generate the same topology on X.

Proof. Using the displayed identity and the strict diagonal assumption, we have

$$-\theta\, \sigma_\varrho(x,y) \le -\delta_\varrho(x) + \delta_\varrho(y) \le \theta\, \sigma_\varrho(x,y).$$

Substitution into the formula for $\eta_\varrho(x,y)$ yields the two-sided estimate. The topological conclusion follows directly from this estimate. $\square$

Remark 2.1. *In the sequel, lower semicontinuity with respect to the partial metric topology may be equivalently understood as lower semicontinuity with respect to the metric topology of σ_ϱ.*

3. A variational principle

We now state the basic variational principle in the strictly diagonal setting.

Theorem 3.1 (Ekeland-type variational principle)**.** *Let (X,ϱ) be a strictly diagonal partial metric space with constant $\theta \in [0,1)$, and assume that (X,σ_ϱ) is complete. Let*

$$\Phi : X \to (-\infty, +\infty]$$

be proper, bounded from below, and lower semicontinuous. Fix $\varepsilon > 0$, $\lambda > 0$, and $x_0 \in X$ such that

$$\Phi(x_0) \le \inf_X \Phi + \varepsilon.$$

Then, there exists $z \in X$ satisfying

$$\Phi(z) \le \Phi(x_0), \qquad \sigma_\varrho(z, x_0) \le \lambda,$$

and

$$\Phi(x) > \Phi(z) - \frac{\varepsilon}{\lambda}\sigma_\varrho(z,x), \qquad x \in X \setminus \{z\}.$$

Proof. By Proposition 2.1, the topology induced by η_ϱ agrees with the metric topology of σ_ϱ. Hence Φ is lower semicontinuous on the complete metric space (X, σ_ϱ). The classical Ekeland variational principle [2] applied in this complete metric space gives the required point z. □

Corollary 3.1 (Quasi-metric formulation)**.** *Under the assumptions of Theorem 3.1, there exists $z \in X$ such that*

$$\Phi(z) \leq \Phi(x_0), \qquad \eta_\varrho(z, x_0) \leq \frac{1+\theta}{2}\lambda,$$

and

$$\Phi(x) > \Phi(z) - \frac{2\varepsilon}{(1-\theta)\lambda}\eta_\varrho(z, x), \qquad x \in X \setminus \{z\}.$$

Proof. The estimate for $\eta_\varrho(z, x_0)$ follows from Proposition 2.1. Moreover,

$$\sigma_\varrho(z, x) \leq \frac{2}{1-\theta}\eta_\varrho(z, x),$$

and the desired inequality follows from Theorem 3.1. □

Proposition 3.1 (Penalized strict minimizer)**.** *Under the assumptions of Theorem 3.1, the point z may be chosen so that the functional*

$$\Psi_z(x) = \Phi(x) + \frac{\varepsilon}{\lambda}\sigma_\varrho(z, x)$$

has a strict global minimum at z.

Proof. For $x \neq z$, Theorem 3.1 gives

$$\Phi(x) + \frac{\varepsilon}{\lambda}\sigma_\varrho(z, x) > \Phi(z).$$

Since $\sigma_\varrho(z, z) = 0$, we have $\Psi_z(x) > \Psi_z(z)$ for every $x \neq z$. □

Proposition 3.2 (Localized version)**.** *Let $A \subset X$ be nonempty and closed with respect to the topology of η_ϱ. If $\Phi|_A$ is proper, bounded below, and lower semicontinuous, then Theorem 3.1 remains valid with X replaced by A.*

Proof. The topology on A is the topology induced by σ_ϱ. Since A is closed in the complete metric space (X, σ_ϱ), it is complete. Applying Theorem 3.1 to A proves the claim. □

4. Fixed point, minimization, and dynamical applications

4.1. Caristi-type and Takahashi-type consequences

Theorem 4.1 (Caristi-type fixed point theorem)**.** *Let (X, ϱ) be a strictly diagonal partial metric space such that (X, σ_ϱ) is complete. Let $\Phi : X \to \mathbb{R}$ be lower semicontinuous and bounded from below. Suppose that a self-map $T : X \to X$ satisfies*

$$\sigma_\varrho(x, Tx) \leq \Phi(x) - \Phi(Tx), \qquad x \in X.$$

Then, T has a fixed point.

Proof. The condition is exactly Caristi's condition on the complete metric space (X, σ_ϱ). The classical Caristi fixed point theorem [1] therefore gives a point $z \in X$ such that $Tz = z$. $\square$

Corollary 4.1 (Quasi-metric Caristi condition)**.** *Under the assumptions of Theorem 4.1, if*

$$\frac{2}{1-\theta}\eta_\varrho(x, Tx) \le \Phi(x) - \Phi(Tx), \qquad x \in X,$$

then T has a fixed point.

Proof. By Proposition 2.1,

$$\sigma_\varrho(x, Tx) \le \frac{2}{1-\theta}\eta_\varrho(x, Tx).$$

Thus the metric Caristi condition of Theorem 4.1 is satisfied. $\square$

Theorem 4.2 (Takahashi-type minimization theorem)**.** *Let (X, ϱ) be a strictly diagonal partial metric space such that (X, σ_ϱ) is complete. Let $\Phi : X \to (-\infty, +\infty]$ be proper, bounded from below, and lower semicontinuous. Assume that for every $x \in X$ satisfying $\Phi(x) > \inf_X \Phi$ there exists $y \in X \setminus \{x\}$ such that*

$$\Phi(y) + \sigma_\varrho(x, y) \le \Phi(x).$$

Then, Φ attains its infimum on X.

Proof. Assume that Φ does not attain its infimum. Choose $\varepsilon \in (0, 1)$ and $x_0 \in X$ such that $\Phi(x_0) \le \inf_X \Phi + \varepsilon$. Applying Theorem 3.1 with $\lambda = 1$, we obtain $z \in X$ such that

$$\Phi(x) > \Phi(z) - \varepsilon\sigma_\varrho(z, x), \qquad x \ne z.$$

Since z is not a minimizer, the hypothesis gives $y \ne z$ with $\Phi(y) + \sigma_\varrho(z, y) \le \Phi(z)$. Hence

$$\Phi(y) \le \Phi(z) - \sigma_\varrho(z, y) < \Phi(z) - \varepsilon\sigma_\varrho(z, y),$$

which contradicts the Ekeland inequality. Therefore Φ attains its infimum. $\square$

4.2. Semigroup and resolvent-type applications

Theorem 4.3 (Common fixed point for a semigroup)**.** *Let (X, ϱ) be a strictly diagonal partial metric space such that (X, σ_ϱ) is complete. Let $\{S(t)\}_{t \ge 0}$ be a semigroup of self-maps on X. Assume that there exists a lower semicontinuous function $\Phi : X \to \mathbb{R}$, bounded from below, such that*

$$\sigma_\varrho(x, S(t)x) \le \Phi(x) - \Phi(S(t)x), \qquad x \in X, \quad t \ge 0.$$

Suppose also that, for some $t_0 > 0$, the map $S(t_0)$ is a strict contraction on (X, σ_ϱ). Then, the semigroup $\{S(t)\}_{t \ge 0}$ has a unique common fixed point.

Proof. By Theorem 4.1, $S(t_0)$ has a fixed point. Since $S(t_0)$ is a strict contraction on the complete metric space (X, σ_ϱ), the fixed point is unique; denote it by z. For any $t \geq 0$,

$$S(t_0)(S(t)z) = S(t)(S(t_0)z) = S(t)z.$$

Thus $S(t)z$ is also a fixed point of $S(t_0)$. By uniqueness, $S(t)z = z$. Hence z is a common fixed point, and its uniqueness is inherited from $S(t_0)$. □

Theorem 4.4 (Commuting resolvent-type family)**.** *Let (X, ϱ) be a strictly diagonal partial metric space such that (X, σ_ϱ) is complete. Let $\{J_\lambda\}_{\lambda>0}$ be a family of self-maps on X satisfying*

$$J_\lambda J_\mu = J_\mu J_\lambda, \qquad \lambda, \mu > 0.$$

Assume that there exists a lower semicontinuous function $\Phi : X \to \mathbb{R}$, bounded from below, such that

$$\sigma_\varrho(x, J_\lambda x) \leq \lambda\Big(\Phi(x) - \Phi(J_\lambda x)\Big), \qquad x \in X, \quad \lambda > 0.$$

If, for some $\lambda_0 > 0$, the map J_{λ_0} is a strict contraction on (X, σ_ϱ), then the family $\{J_\lambda\}_{\lambda>0}$ has a unique common fixed point.

Proof. Apply Theorem 4.1 to J_{λ_0} and to the potential $\lambda_0\Phi$. The map J_{λ_0} has a fixed point, and the strict contraction assumption makes it unique; call it z. For $\mu > 0$, commutativity gives

$$J_{\lambda_0}(J_\mu z) = J_\mu(J_{\lambda_0} z) = J_\mu z.$$

Thus $J_\mu z$ is a fixed point of J_{λ_0}, so $J_\mu z = z$. Hence z is a common fixed point of the whole family. □

5. **Concrete models**

Example 5.1 (A basic strictly diagonal model)**.** *Let $X = [0, \infty)$ and fix $a \in [0, 1)$. Define*

$$\varrho(x, y) = \frac{1}{2}\Big(|x - y| + ax + ay\Big), \qquad x, y \in X.$$

Then,

$$\delta_\varrho(x) = ax, \qquad \sigma_\varrho(x, y) = |x - y|, \qquad \eta_\varrho(x, y) = \frac{1}{2}\Big(|x - y| - ax + ay\Big).$$

Since $x \mapsto ax$ is a-Lipschitz on $(X, |\cdot|)$, the space is strictly diagonal with constant $\theta = a$. Moreover, σ_ϱ is the Euclidean distance on $[0, \infty)$.

Example 5.2 (Penalized strict minimizer)**.** *Consider the previous model with $a = 0.4$ and define*

$$\Phi(x) = (x - 2)^2 + 1, \qquad x \in X.$$

Here $\inf_X \Phi = 1$, *attained at* $x = 2$. *Let* $x_0 = 3$, $\varepsilon = 2$, *and* $\lambda = 2$. *Then,*

$$\Phi(3) = 2 \le \inf_X \Phi + \varepsilon.$$

Theorem 3.1 applies and one may select $z = 2$, *because* $\sigma_\varrho(2,3) = 1 \le 2$ *and* $\Phi(2) = 1 \le \Phi(3)$. *The penalized functional is*

$$\Psi_2(x) = (x-2)^2 + 1 + |x-2|,$$

which has a strict global minimum at $x = 2$.

Example 5.3 (A Caristi map with explicit iteration)**.** *Let* $a = 0.4$, $\Phi(x) = x$, *and*

$$T(x) = \frac{x}{1+x}, \qquad x \in X.$$

Then,

$$\sigma_\varrho(x, Tx) = \left| x - \frac{x}{1+x} \right| = \frac{x^2}{1+x}$$

and

$$\Phi(x) - \Phi(Tx) = x - \frac{x}{1+x} = \frac{x^2}{1+x}.$$

Thus the Caristi condition holds with equality, and the fixed point is 0. *Starting from* $x_0 = 4$, *the Picard sequence* $x_{n+1} = T(x_n)$ *begins as*

$$x_1 = 0.8, \quad x_2 = 0.4444, \quad x_3 \simeq 0.3077, \quad x_4 \simeq 0.2353,$$

and it converges monotonically to 0.

Example 5.4 (A semigroup model)**.** *Let* $X = [0, \infty)$ *and let* ϱ *be the partial metric of the first example with* $a = 0.3$. *Define*

$$S(t)x = 2 + e^{-t}(x-2), \qquad t \ge 0, \quad x \in X.$$

Then, $S(0) = I$ *and* $S(t+s) = S(t)S(s)$. *With* $\Phi(x) = |x-2|$, *we have*

$$\Phi(S(t)x) = e^{-t}|x-2|$$

and

$$\sigma_\varrho(x, S(t)x) = (1 - e^{-t})|x-2| = \Phi(x) - \Phi(S(t)x).$$

Moreover $S(t)$ *is a strict contraction for* $t > 0$, *with contraction constant* e^{-t}. *Hence the semigroup has the unique common fixed point* 2. *For* $x = 8$,

$$S(1)8 \simeq 4.2073, \quad S(2)8 \simeq 2.8120, \quad S(3)8 \simeq 2.2987.$$

Example 5.5 (A commuting resolvent-type family)**.** *Let* $a = 0.2$ *and define, for* $\lambda > 0$,

$$J_\lambda(x) = 2 + \frac{x-2}{1+\lambda} = \frac{x+2\lambda}{1+\lambda}.$$

Then, $J_\lambda J_\mu = J_\mu J_\lambda$. *If* $\Phi(x) = |x-2|$, *then*

$$\sigma_\varrho(x, J_\lambda x) = \frac{\lambda}{1+\lambda}|x-2|$$

and

$$\lambda\Big(\Phi(x) - \Phi(J_\lambda x)\Big) = \frac{\lambda^2}{1+\lambda}|x-2|.$$

Thus the resolvent inequality holds for $\lambda \geq 1$. Restricted to $\lambda \in [1,\infty)$, each J_λ is a strict contraction and the common fixed point is 2. For $\lambda = 1$ and $x_0 = 8$,

$$x_1 = 5, \quad x_2 = 3.5, \quad x_3 = 2.75, \quad x_4 = 2.375,$$

so the distance to equilibrium is halved at each step.

Remark 5.1. *The examples show that the abstract variational principle applies not only to individual maps but also to parameter-dependent families and dynamical systems. The partial metric structure is controlled by the strict diagonal condition, while the effective convergence analysis is performed in the symmetrized metric model.*

6. Set-valued, asymmetric, and evolutionary schemes

6.1. A multivalued variational principle on closed graphs

Let $F : X \rightrightarrows X$ be a nonempty-valued mapping and denote its graph by

$$\mathrm{Gr}(F) = \{(x,y) \in X \times X : y \in F(x)\}.$$

On $X \times X$, consider the product metric

$$\Sigma((x,y),(r,s)) = \sigma_\varrho(x,r) + \sigma_\varrho(y,s).$$

Since (X, σ_ϱ) is complete, the product space $(X \times X, \Sigma)$ is complete.

Theorem 6.1 (Set-valued variational principle)**.** *Let (X,ϱ) be a strictly diagonal partial metric space such that (X,σ_ϱ) is complete. Let $F : X \rightrightarrows X$ have nonempty values and assume that $\mathrm{Gr}(F)$ is closed in $X \times X$ for the product topology induced by Σ. Let*

$$\Psi : \mathrm{Gr}(F) \to (-\infty, +\infty]$$

be proper, bounded from below, and lower semicontinuous. Fix $\varepsilon > 0$, $\lambda > 0$, and $(x_0, y_0) \in \mathrm{Gr}(F)$ such that

$$\Psi(x_0,y_0) \leq \inf_{\mathrm{Gr}(F)} \Psi + \varepsilon.$$

Then, there exists $(z,w) \in \mathrm{Gr}(F)$ such that

$$\Psi(z,w) \leq \Psi(x_0,y_0), \qquad \Sigma((z,w),(x_0,y_0)) \leq \lambda,$$

and

$$\Psi(x,y) > \Psi(z,w) - \frac{\varepsilon}{\lambda}\Sigma((z,w),(x,y))$$

for every $(x,y) \in \mathrm{Gr}(F) \setminus \{(z,w)\}$.

Proof. The graph is closed in the complete metric space $(X \times X, \Sigma)$ and is therefore complete. Applying the classical Ekeland variational principle to Ψ on $\mathrm{Gr}(F)$ gives the conclusion. $\square$

Corollary 6.1 (A graph-based strict minimizer)**.** *Under the assumptions of Theorem 6.1, the pair (z, w) may be chosen so that*

$$(x, y) \mapsto \Psi(x, y) + \frac{\varepsilon}{\lambda}\Sigma((z, w), (x, y))$$

has a strict global minimum at (z, w) on $\mathrm{Gr}(F)$.

Example 6.1 (A compact-valued graph model)**.** *In the model of Section 5 with $a = 0.3$, set*

$$F(x) = [2, 2 + 0.2|x - 2|], \qquad x \in [0, \infty).$$

Its graph is closed. For

$$\Psi(x, y) = |y - 2| + \frac{1}{2}|x - y|,$$

we have $\Psi \geq 0$ and $\Psi(2, 2) = 0$. Thus the optimal variational pair is $(2, 2)$.

6.2. Equilibrium problems in asymmetric settings

The quasi-metric η_ϱ leads to asymmetric transition costs.

Definition 6.1. *An asymmetric transition kernel on X is a mapping $A : X \times X \to [0, \infty)$ such that $A(x, x) = 0$ for all $x \in X$. It is comparable with η_ϱ if there exist constants $0 < m \leq M$ such that*

$$m\eta_\varrho(x, y) \leq A(x, y) \leq M\eta_\varrho(x, y), \qquad x, y \in X.$$

Theorem 6.2 (Asymmetric equilibrium principle)**.** *Let (X, ϱ) be a strictly diagonal partial metric space such that (X, σ_ϱ) is complete. Let A be an asymmetric transition kernel comparable with η_ϱ through constants $0 < m \leq M$. Let $\Phi : X \to (-\infty, +\infty]$ be proper, bounded from below, and lower semicontinuous. If $\Phi(x_0) \leq \inf_X \Phi + \varepsilon$, then there exists $z \in X$ such that*

$$A(z, x_0) \leq \frac{M(1 + \theta)}{2}\lambda$$

and

$$\Phi(z) \leq \Phi(y) + \frac{2\varepsilon}{m(1 - \theta)\lambda}A(z, y), \qquad y \in X.$$

Proof. By Corollary 3.1, there exists $z \in X$ such that

$$\eta_\varrho(z, x_0) \leq \frac{1 + \theta}{2}\lambda$$

and

$$\Phi(y) > \Phi(z) - \frac{2\varepsilon}{(1 - \theta)\lambda}\eta_\varrho(z, y) \qquad (y \neq z).$$

Using $A(z,x_0) \le M\eta_\varrho(z,x_0)$ and $\eta_\varrho(z,y) \le A(z,y)/m$ gives the desired estimates. The case $y=z$ is trivial because $A(z,z)=0$. □

Example 6.2 (An asymmetric transition cost)**.** *In the model of Section 5 with* $a = 0.4$*, define*

$$A(x,y) = \eta_\varrho(x,y) + 0.2|x-y|.$$

By Proposition 2.1, A is comparable with η_ϱ. For $\Phi(x) = (x-1)^2$ and an approximate minimizer $x_0 = 2$, Theorem 6.2 yields an equilibrium inequality of the form

$$\Phi(z) \le \Phi(y) + CA(z,y), \qquad y \in X,$$

where C is explicit in ε, λ, and the comparability constants.

6.3. **Variational generation schemes**

Let $\phi : X \to (-\infty,+\infty]$ be proper and lower semicontinuous. For $h>0$ and $u \in X$, define

$$F_{h,u}(v) = \phi(v) + \frac{1}{2h}\sigma_\varrho(v,u)^2, \qquad v \in X.$$

A minimizer of $F_{h,u}$ is called a variational resolvent step from u with time increment h.

Proposition 6.1 (Existence of one-step minimizers)**.** *Let (X,ϱ) be a strictly diagonal partial metric space such that (X,σ_ϱ) is complete, and let $\phi : X \to (-\infty,+\infty]$ be proper, bounded from below, and lower semicontinuous. Then, for every $h>0$ and every $u \in X$, the functional $F_{h,u}$ attains its minimum on X.*

Proof. The map $v \mapsto \sigma_\varrho(v,u)^2$ is continuous on (X,σ_ϱ), so $F_{h,u}$ is proper, bounded from below, and lower semicontinuous. The Takahashi-type minimization argument [6] applied to (X,σ_ϱ) gives the existence of a minimizer. □

Theorem 6.3 (Implicit variational generation scheme)**.** *Under the assumptions of Proposition 6.1, fix $u_0 \in \mathrm{dom}(\phi)$ and $h>0$. Then, there exists a sequence $\{u_h^n\}_{n\ge 0}$ such that $u_h^0 = u_0$ and*

$$u_h^n \in \operatorname*{argmin}_{v\in X}\left\{\phi(v) + \frac{1}{2h}\sigma_\varrho(v,u_h^{n-1})^2\right\}, \qquad n \ge 1.$$

Moreover,

$$\phi(u_h^n) + \frac{1}{2h}\sigma_\varrho(u_h^n,u_h^{n-1})^2 \le \phi(u_h^{n-1}), \qquad n \ge 1.$$

Consequently,

$$\phi(u_h^n) \le \phi(u_h^{n-1}) \le \cdots \le \phi(u_0)$$

and

$$\sum_{k=1}^{n} \sigma_\varrho(u_h^k,u_h^{k-1})^2 \le 2h\Big(\phi(u_0) - \inf_X \phi\Big).$$

Proof. The existence of each step follows inductively from Proposition 6.1. Since u_h^n minimizes $F_{h,u_h^{n-1}}$, we have

$$\phi(u_h^n) + \frac{1}{2h}\sigma_\varrho(u_h^n, u_h^{n-1})^2 \leq \phi(u_h^{n-1}).$$

This proves the discrete energy inequality. Summing it for $k = 1, \dots, n$ yields the cumulative estimate. □

Example 6.3 (A quadratic evolution model)**.** *In the model of Section 5 with* $a = 0.3$*, take*

$$\phi(x) = \frac{1}{2}(x-2)^2.$$

Then,

$$F_{h,u}(v) = \frac{1}{2}(v-2)^2 + \frac{1}{2h}(v-u)^2$$

is strictly convex on $[0, \infty)$ *and has the unique minimizer*

$$J_h(u) = \frac{u+2h}{1+h} = 2 + \frac{u-2}{1+h}.$$

The discrete evolution is

$$u_h^n = 2 + \frac{u_0 - 2}{(1+h)^n}.$$

For $u_0 = 8$ *and* $h = 1$*, one obtains*

$$u_1 = 5, \quad u_2 = 3.5, \quad u_3 = 2.75, \quad u_4 = 2.375.$$

These values coincide with the backward-Euler approximation of

$$u'(t) + u(t) - 2 = 0, \qquad u(0) = 8,$$

whose exact solution is $u(t) = 2 + 6e^{-t}$*.*

7. Concluding remarks

We have developed an Ekeland-type variational principle for strictly diagonal partial metric spaces. The main assumption gives quantitative control of the topology and allows variational arguments to be transferred from the symmetrized metric setting to the partial metric framework.

The resulting theory yields a penalized strict minimization principle, Caristi-type and Takahashi-type consequences, common fixed point theorems for semigroups and commuting resolvent-type families, and several extensions to set-valued, asymmetric, and evolutionary settings. The examples demonstrate that the abstract framework can be implemented in explicit models with transparent convergence estimates.

Further work may address sharper compactness conditions for multivalued mappings, refined equilibrium conditions for nonsymmetric bifunctions, and convergence of continuous-time limits for the variational generation scheme in broader classes of partial metric spaces.

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