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A FAMILY OF GEOMETRIC CONSTANTS AND THE FIXED POINT PROPERTY FOR MULTIVALUED NON-EXPANSIVE MAPPINGS

Chao XIA[1], Junqing YIN[2], Xi WANG[3*]

The existence of fixed points for multivalued nonexpansive mappings is closely related to the geometric structure of the underlying Banach space. In this paper, sufficient conditions are established on a Banach space X in terms of the generalized geometric constant $C_{\lambda,t}(X)$, the asymptotic coefficient $R(1,X)$, and the weak orthogonality coefficient $\mu(X)$. It is shown that a sufficiently small value of $C_{\lambda,t}(X)$ implies the (DL)- condition. Consequently, the existence of fixed points is guaranteed for multivalued non-expansive self-mappings defined on weakly compact convex subsets of X. These results extend several classical fixed point theorems, illuminating normal structure and the fixed point property in geometric settings.

Keywords: multivalued nonexpansive mapping, fixed point, a family of geometric constants, generalized García-Falset coefficient, the coefficient of weak orthogonality

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1. Introduction

In 1969, Nadler [19] established the multivalued version of Banach contraction principle. A key technology to get fixed point property for multivalued nonexpansive mapping is Edelstein's method of asymptotic centers. For instance, using it T. C. Lim [18] proved that every multivalued non-expansive self-mapping $T : E \to K(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a uniformly convex Banach space X. W. A. Kirk and S. Massa [17] proved that if a nonempty bounded closed convex subset E of a Banach space X has a property that the asymptotic center in E of

[1]School of Mathematics and Statistics, Changchun University of Technology, Changchun 130012, P.R.China, e-mail: `xiachaomath@gmail.com`

[2]School of Mathematics and Statistics, Changchun University of Technology, Changchun 130012, P.R.China, e-mail: `yinjunqing2004@163.com`

[3]School of Mathematics and Statistics, Changchun University of Technology, Changchun 130012, P.R.China, e-mail: `wangxi@ccut.edu.cn`

each bounded sequence of X is nonempty and compact, then every multivalued non-expansive self-mapping $T : E \to KC(E)$ has a fixed point. In 2004, Domínguez and Lorenzo [5] proved that every multivalued non-expansive mapping $T : E \to KC(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a nearly uniformly convex Banach space X.

In 2006, S. Dhompongsa et al. [9, 10] introduced the Domínguez-Lorenzo condition and property (D) which imply the fixed point property for multivalued nonexpansive mappings. In 2007, T. D. Benavides and Gavira [3] had established the fixed point property for multivalued nonexpansive mappings in terms of the modulus of squareness, universal infinite-dimensional modulus, and Opial modulus. A. Kaewkhao [15] has established the fixed point property for multivalued nonexpansive mappings in terms of the James constant, the Jordan-von Neumann constant, weak orthogonality. In 2010, T. D. Benavides and Gavira [4] had given a survey of this subject and presented the main known results and current research directions. For more details about recent work on fixed point property for multivalued nonexpansive mapping, one can refer to [21, 22, 23, 24].

Let X be a Banach space with unit ball $B_X = \{x \in X : \|x\| \leq 1\}$ and unit sphere $S_X = \{x \in X : \|x\| = 1\}$. The following constant of a Banach space

$$C_{NJ}(X) = \sup\left\{\frac{\| x+y \|^2 + \| x-y \|^2}{2(\| x \|^2 + \| y \|^2)} : x, y \in X, (x,y) \neq (0,0)\right\}.$$

is called the von Neumann-Jordan constant [6], which is widely studied by many authors [19, 9, 3, 4].

In order to promote the results of $C_{NJ}(X)$, Dhompangsa in [8] introduced the constant $C_{NJ}(a, X)$, for $a \geq 0$.

$$C_{NJ}(a, X) = \sup\left\{\frac{\| x+y \|^2 + \| x-z \|^2}{2 \| x \|^2 + \| y \|^2 + \|z\|^2} : x, y, z \in X\right\},$$

where $(x, y, z) \neq (0, 0, 0)$, and $\|y - z\| \leq a\|x\|$. It is clear that $C_{NJ}(0, X) = C_{NJ}(X)$.

Cui [7] and Dinarvand [11] introduced the constant $C_{NJ}^{(p)}(X)$ and $C_{NJ}^{(p)}(a, X)$, respectively, and gave some sufficient conditions for the normal structure, where $a \geq 0, 1 \leq p < \infty$.

$$C_{NJ}^{(p)}(X) = \sup\left\{\frac{\| x+y \|^p + \| x-y \|^p}{2^{p-1}(\| x \|^p + \| y \|^p)} : x, y \in X, (x,y) \neq (0,0)\right\}.$$

$$C_{NJ}^{(p)}(a,X)=\sup\left\{\frac{\| x+y \|^p+\| x-z \|^p}{2^{p-2}\| x \|^p+2^{p-3}(\| y \|^p+\|z\|^p)}:x,y,z\in X\right\}$$

where $(x,y,z)\neq(0,0,0)$, and $\|y-z\|\leq a\|x\|$. It was proved that the generalized von Neumann-Jordan constant satisfies the inequality $C_{NJ}^{(p)}(X)\leq 2$, and that Banach space X is uniformly non-square if and only if $C_{NJ}^{(p)}(X)<2$ (see [7]). If $C_{NJ}^{(p)}(X)<1+\frac{1}{\mu(X)^p}$, then the Banach space X has normal structure (see [20]).

In 2020, A. Amini-Harandi and M. Rahimi defined a new constant [1]

$$C_{\lambda,t}=\sup\{\lambda(\|ru+sv\|,\|ru-sv\|):u,v\in S_X,r,s\geq 0,\|(r,s)\|_t=1\},$$

where $1\leq t<\infty$, X is a real norm space,

$$\|(r,s)\|_t=\begin{cases}(|r|^t+|s|^t)^{\frac{1}{t}} & \text{if } 1\leq p<\infty\\ \max\{|r|,|s|\} & \text{if } p=\infty\end{cases}$$

and λ is a continuously homogeneous of degree 1 function from $[0,\infty)\times[0,\infty)$ to $[0,\infty)$. This constant $C_{\lambda,t}(X)$ generalize some well-known geometric constants, such as James constant $J(X)$, Jordan-von Neumann constant $C_{NJ}(X)$, generalized Jordan-von Neumann constant $C_{NJ}^{(p)}(X)$, Zbăganu constant $C_Z(X)$ and so on. More properties of $C_{\lambda,t}(X)$ can be found in [12].

In this paper, we discuss the constant $C_{\lambda,t}$ with the fixed point property for multivalued non-expansive mappings. The constant $C_{\lambda,t}(X)$ measures the squareness of the unit ball of X in a generalized sense. While classical constants such as the James constant $J(X)$ or the von Neumann-Jordan constant $C_{NJ}(X)$ capture symmetry along specific axes, $C_{\lambda,t}(X)$ quantifies the deviation of the norm from being induced by an inner product for arbitrary pairs of vectors scaled by parameters r and s. Intuitively, a smaller value of $C_{\lambda,t}(X)$ indicates that the space is closer to being uniformly non-square. This property is crucial for ensuring the asymptotic center of a bounded sequence remains compact, thereby guaranteeing the fixed point property.

The remainder of this paper is organized as follows. In Section 2 we recall basic definitions and auxiliary results concerning multivalued nonexpansive mappings and geometric coefficients. Section 3 is devoted to establishing a sufficient condition for the fixed point property in terms of the coefficient $R(1,X)$ and the constant $C_{\lambda,t}(X)$. Section 4 we obtain a parallel result involving the weak orthogonality coefficient $\mu(X)$. Section 5 presents a comparative discussion of our results with existing literature and puts the proposed criteria into perspective. Finally, Section 6 summarizes the main findings.

2. **Preliminaries**

The following coefficient is defined by T. D. Benavides [2] as

$$R(1,X)=\sup\{\liminf_{n\to\infty}\parallel x_n+x\parallel\},$$

where the supremum is taken over all $x\in X$ with $\parallel x\parallel\leq 1$ and all weakly null sequences (x_n) in the unit ball B_X such that

$$D[(x_n)]:=\limsup_{n\to\infty}(\limsup_{m\to\infty}\parallel x_n-x_m\parallel)\leq 1.$$

It is clear that $1\leq R(1,X)\leq 2$. Some geometric condition sufficient for normal structure in term of this coefficient have been studied in [13, 20].

The coefficient of weak orthogonality $\mu(X)$, defined by the infmum of the set of real numbers $\lambda>0$ such that

$$\limsup_{n\to\infty}\parallel x+x_n\parallel\leq\lambda\limsup_{n\to\infty}\parallel x-x_n\parallel$$

for all $x\in X$ and all weakly null sequences (x_n) in X [14].

Let C be a nonempty subset of a Banach space X. We shall denote by $CB(X)$ the family of all nonempty closed bounded subsets of X and by $KC(X)$ the family of all nonempty compact convex subsets of X . A multivalued mapping $T:C\to CB(X)$ is said to be nonexpansive if

$$H(Tx,Ty)\leq\parallel x-y\parallel,$$

for all $x,y\in C$, where $H(.,.)$ denotes the Hausdorff metric on $CB(X)$ defined by

$$H(A,B):=\max\{\sup_{x\in A}\inf_{y\in B}\parallel x-y\parallel,\sup_{y\in B}\inf_{x\in A}\parallel x-y\parallel\},A,B\in CB(X).$$

Let $\{x_n\}$ be a bounded sequence in X. The asymptotic radius $r(C,\{x_n\})$ and the asymptotic center $A(C,\{x_n\})$ of $\{x_n\}$ in C are defined by

$$r(C,\{x_n\})=\inf\{\limsup_{n}\parallel x_n-x\parallel x\in C\}$$

and

$$A(C,\{x_n\})=\{x\in C:\limsup_{n}\parallel x_n-x\parallel=r(C,\{x_n\})\},$$

respectively. It is known that $A(C,\{x_n\})$ is a nonempty weakly compact convex set whenever C is. The sequence $\{x_n\}$ is called regular with respect to C if $r(C,\{x_n\})=r(C,\{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$, and $\{x_n\}$ is called asymptotically uniform with respect to C if $A(C,\{x_n\})=A(C,\{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$. If D is a bounded subset of X , the Chebyshev radius of D relative to C is defined by

$$r_C(D)=\inf_{x\in C}\sup_{y\in D}\parallel x-y\parallel.$$

S. Dhompongsa et al.[10] introduced the property (D) if there exists $\lambda \in [0,1)$ such that for any nonempty weakly compact convex subset C of X, any sequence $\{x_n\} \subset C$ which is regular asymptotically uniform relative to C, and any sequence $\{y_n\} \subset A(C, \{x_n\})$ which is regular asymptotically uniform relative to X we have

$$r(C, \{y_n\}) \leq \lambda r(C, \{x_n\}).$$

The Domínguez-Lorenzo condition ((DL)-condition,in short) introduced in [9] is defined as follows: if there exists $\lambda \in [0,1)$ such that for every weakly compact convex subset C of X and for every bounded sequence $\{x_n\}$ in C which is regular with respect to C,

$$r_C(A(C, \{x_n\})) \leq \lambda r(C, \{x_n\}).$$

It is clear from the definition that property (D) is weaker than the (DL)-condition. The next results shows that property (D) is stronger than weak normal structure and also implies the existence of fixed points for multivalued nonexpansive mappings [10]: Let X be a Banach space satisfying ((DL)-condition) property (D), Then X has weak normal structure; Let C be a nonempty weakly compact convex subset of a Banach space X which satisfies ((DL)-condition) the property (D). Let $T : C \to KC(C)$ be a multivalued nonexpansive mapping, then T has a fixed point.

3. **The constant $C_{\lambda,t}$ and the coefficient $R(1, X)$**

In this section, we show a sufficient condition concerning the generalized von Neumann-Jordan constant, and the coefficient $R(1, X)$, which implies the existence of fixed points for multivalued nonexpansive mappings.

First recall some basic facts about ultrapowers. Let $\mathcal{F}$ be a filter on $\mathbb{N}$. A sequence $\{x_n\}$ in X converges to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_n = x$ if for each neighborhood U of x, $\{n \in \mathbb{N}\} \in \mathcal{F}$. A filter U on $\mathbb{N}$ is called to be an ultrafilter if it is maximal with respect to set inclusion. An ultrafilter is called trivial if it is of the form $A : A \in \mathbb{N}$, $n_0 \in A$ for some fixed $n_0 \in \mathbb{N}$, otherwise, it is called nontrivial. let $l_\infty(X)$ denotes that the subspace of the product space $\Pi_{n\in\mathbb{N}} X$ equipped with the norm $\| (x_n) \| := \sup_{n\in\mathbb{N}} \| x_n \| < \infty$. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and let

$$N_{\mathcal{U}} = \{(x_n) \in l_\infty(X) : \lim_{\mathcal{U}} \| x_n \| = 0\}.$$

The ultrapower of X, denoted by $\tilde{X}$, is the quotient space $l_\infty(X)/N_{\mathcal{U}}$ equipped with the quotient norm, and $(x_n)_{\mathcal{U}}$ denotes the elements of the ultrapower. Note that if $\mathcal{U}$ is non-trivial, then X can be embedded into $\tilde{X}$

isometrically. It was shown that if the space X is super-reflexive, then X has uniformly structure if and if $\tilde{X}$ has normal structure (see [16]).

Theorem 3.1 (Main)**.** *Let C be a weakly compact convex subset of a Banach space X and $\{x_n\}$ is a bounded sequence in C regular with respect to C. Suppose that λ is a continuously homogeneous of degree 1 function from $[0,\infty)\times[0,\infty)$ to $[0,\infty)$ and non-decreasing in both variable. Then, we have*

$$r_C(A(C,\{x_n\})) \leq \frac{2^{\frac{1}{t}}R(1,X)C_{\lambda,t}(X)}{(R(1,X)+1)}r(C,\{x_n\}).$$

Proof. Denote $r(C,\{x_n\})$ as r and $A(C,\{x_n\})$ as A. We should assume that $r>0$, by passing to a subsequence if necessary, we can also assume that $\{x_n\}$ is weakly convergent to a point $x\in C$ and $d=\lim_{n\neq m}\parallel x_n - x_m\parallel$ exists. Since $\{x_n\}$ is regular with respect to C, passing through a subsequence does not have any effect to the asymptotic radius of the whole sequence $\{x_n\}$. Observe that the norm is weakly lower semicontinuous, we have

$$\liminf_n \parallel x_n - x\parallel \leq \liminf_n \liminf_m \parallel x_n - x_m\parallel = \lim_{n\neq m}\parallel x_n - x_m\parallel = d.$$

Let $\varepsilon>0$. Taking a subsequence if necessary, we can assume that $\parallel x_n - x\parallel < d+\varepsilon$ for all n. Let $z\in A$, then we have $\limsup_n \parallel x_n - z\parallel = r$ and $\parallel x - z\parallel \leq \liminf_n \parallel x_n - z\parallel \leq r$. Denote $R=R(1,X)$, then by definition we have

$$R \geq \liminf_n \left\|\frac{x_n - x}{d+\varepsilon} + \frac{z-x}{r}\right\| = \liminf_n \left\|\frac{x_n - x}{d+\varepsilon} - \frac{z-x}{r}\right\|.$$

Since C is convex, the point $\frac{R-1}{R+1}x + \frac{2}{R+1}z$ belongs to C. Moreover, the norm is weakly lower semicontinuous. Combining these facts, we estimate the asymptotic radius as follows:

$$\begin{aligned}
&\liminf_n \left\|\frac{x_n - z}{r} + \frac{1}{R}\Big(\frac{x_n - x}{d+\varepsilon} - \frac{x-z}{r}\Big)\right\| \\
&= \liminf_n \left\|\Big(\frac{1}{r} + \frac{1}{R(d+\varepsilon)}\Big)(x_n - x) + \Big(\frac{1}{r} - \frac{1}{Rr}\Big)x - \Big(\frac{1}{r} - \frac{1}{Rr}\Big)z\right\| \\
&\geq \left\|\frac{R-1}{Rr}x + \frac{2}{Rr}z - \frac{R+1}{Rr}z\right\| \\
&= \frac{R+1}{Rr}\left\|\frac{R-1}{R+1}x + \frac{2}{R+1}z - z\right\| \\
&\geq \Big(1+\frac{1}{R}\Big)\frac{r_C(A)}{r}.
\end{aligned}$$

Note that the last inequality holds because $z \in A(C, \{x_n\})$. Consequently, we obtain

$$\begin{aligned}
&\liminf_n \Big\| \frac{x_n - z}{r} - \frac{1}{R}\Big(\frac{x_n - x}{d+\varepsilon} - \frac{x-z}{r}\Big)\Big\| \\
&\geq \Big\| \Big(\frac{1}{r} - \frac{1}{R(d+\varepsilon)}\Big)(x_n - x) + \Big(1 + \frac{1}{R}\Big)\frac{x-z}{r}\Big\| \\
&\geq \Big(1 + \frac{1}{R}\Big)\frac{r_C(A)}{r}.
\end{aligned}$$

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

(1) $\| x_N - z \| \leq r + \varepsilon$;

(2) $\Big\| \frac{(x_N - x)}{d+\varepsilon} - \frac{x-z}{r} \Big\| \leq R\Big(\frac{r+\varepsilon}{r}\Big)$;

(3) $\Big\| R(x_N - z) + \frac{r(x_N - x)}{d+\varepsilon} - (x-z) \Big\| \geq (R+1) r_C(A)(\frac{r-\varepsilon}{r})$;

(4) $\Big\| R(x_N - z) - \Big(\frac{r(x_N - x)}{d+\varepsilon} - (x-z)\Big) \Big\| \geq (R+1) r_C(A)(\frac{r-\varepsilon}{r})$.

Now, let $\tilde{u} = (\frac{x_N - z}{r+\varepsilon})_{\mathcal{U}}$, $\tilde{v} = \frac{1}{R(r+\varepsilon)}(\frac{r(x_N-x)}{d+\varepsilon} - (x-z))_{\mathcal{U}}$. Using the above estimates, we obtain that both of $\tilde{u}$, and $\tilde{v}$ belong to B_X. Then,

$$\begin{aligned}
\| \tilde{u} + \tilde{v} \| &= \frac{1}{R(r+\varepsilon)} \Big\| R(x_N - z) + \frac{r(x_N - x)}{d+\varepsilon} - (x-z) \Big\| \\
&\geq \Big(1 + \frac{1}{R}\Big)\frac{r_C(A)}{r}\Big(\frac{r-\varepsilon}{r+\varepsilon}\Big), \\
\| \tilde{u} - \tilde{v} \| &= \frac{1}{R(r+\varepsilon)} \Big\| R(x_N - z) - \Big(\frac{r(x_N - x)}{d+\varepsilon} - (x-z)\Big) \Big\| \\
&\geq \Big(1 + \frac{1}{R}\Big)\frac{r_C(A)}{r}\Big(\frac{r-\varepsilon}{r+\varepsilon}\Big).
\end{aligned}$$

Since λ is nondecreasing in both variable, and by the definition of $C_{\lambda,t}(\tilde{X})$, we have

$$\begin{aligned}
C_{\lambda,t}(\tilde{X}) &\geq \frac{\lambda(\| \tilde{u} + \tilde{v} \|, \| \tilde{u} - \tilde{v} \|)}{(\| \tilde{u} \|^t + \| \tilde{v} \|^t)^{\frac{1}{t}}} \\
&\geq \Big(\frac{1}{2^{\frac{1}{t}}}\frac{r_C(A)}{r}\Big)\Big(\frac{r-\varepsilon}{r+\varepsilon}\Big)\Big(1 + \frac{1}{R}\Big).
\end{aligned}$$

Since the above inequality is true for every $\varepsilon > 0$ and $C_{\lambda,t}(X) = C_{\lambda,t}(\tilde{X})$ (see Lemma 2 in [1]), we obtain that

$$r_C(A(C, \{x_n\})) \leq \frac{2^{\frac{1}{t}} R(1,X) C_{\lambda,t}(X)}{(R(1,X) + 1)} r(C, \{x_n\}).$$

□

Corollary 3.1. *Let C be a nonempty bounded closed convex subset of a Banach space X such that $C_{\lambda,t}(X) < \frac{1}{2^{\frac{1}{t}}}\Big(1 + \frac{1}{R(1,X)}\Big)$ and $T : C \to KC(C)$ be a multivalued nonexpansive mapping, then T has a fixed point.*

Proof. If $C_{\lambda,t}(X) < \frac{1}{2^{\frac{1}{t}}}\Big(1 + \frac{1}{R(1,X)}\Big)$, then X satisfy the (DL)-condition by Theorem 3.1, so T has a fixed point. □

Corollary 3.2. *Let X be a Banach space such that $C_{\lambda,t}(X) < \frac{1}{2^{\frac{1}{t}}}\Big(1 + \frac{1}{R(1,X)}\Big)$, then X has normal structure.*

Proof. By Theorem 3.1, it is easy to prove that X has weak normal structure. Since $1 \le R(1,X) \le 2$, we obtain $C_{\lambda,t}(X) < \frac{1}{2^{\frac{1}{t}}}\Big(1 + \frac{1}{R(1,X)}\Big) < 2$. This implies that X is uniformly nonsquare, then X is reflexive, therefore weakly normal structure coincide with normal structure. □

4. The constant $C_{\lambda,t}$ and the coefficient of weak orthogonality

In this section, we show a sufficient condition concerning the constant $C_{\lambda,t}$, and the coefficient of weak orthogonality, which implies the existence of fixed points for multivalued nonexpansive mappings.

Theorem 4.1. *Let C be a weakly compact convex subset of a Banach space X and $\{x_n\}$ is a bounded sequence in C regular with respect to C. Suppose that λ is a continuously homogeneous of degree 1 function from $[0,\infty) \times [0,\infty)$ to $[0,\infty)$ and non-decreasing in both variable. Then, we have*

$$\lambda\Big(1, \frac{r_C(A(C,\{x_n\}))}{r(C,\{x_n\})}\Big) \le \frac{\mu(\mu^t+1)^{\frac{1}{t}} C_{\lambda,t}(X)}{\mu^2+1}.$$

Proof. Denote $r = r(C,\{x_n\})$, $A = A(C,\{x_n\})$, and $\mu = \mu(X)$. By passing to a subsequence we may assume $r > 0$ and $x_n \rightharpoonup x \in C$. For any $z \in A$ we have $\limsup \|x_n - z\| = r$ and $\|x - z\| \le r$.

Since $\mu(X)$ measures the failure of weak orthogonality,

$$\limsup_{n\to\infty} \|x_n - 2x + z\| \le \limsup_{n\to\infty} \|(x_n - x) - (z - x)\| = \mu r.$$

The convexity of C implies that $\frac{2}{\mu^2+1}x + \frac{\mu^2-1}{\mu^2+1}z \in C$, and by the definition of the asymptotic radius,

$$\limsup_{n\to\infty} \Big\| x_n - \Big(\frac{2}{\mu^2+1}x + \frac{\mu^2-1}{\mu^2+1}z\Big)\Big\| \ge r.$$

By the weak lower semicontinuity of the norm,

$$\liminf_{n\to\infty} \|(\mu^2-1)(x_n - x) - (\mu^2+1)(z - x)\| \ge (\mu^2+1)\|z - x\|.$$

Fix $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that

(1) $\|x_N - z\| \leq r + \varepsilon$,

(2) $\|x_N - 2x + z\| \leq \mu(r + \varepsilon)$,

(3) $\left\| x_N - \dfrac{2}{\mu^2+1}x - \dfrac{\mu^2-1}{\mu^2+1}z \right\| \geq r - \varepsilon$,

(4) $\|(\mu^2-1)(x_N - x) - (\mu^2+1)(z - x)\| \geq (\mu^2+1)\|z - x\| \frac{r-\varepsilon}{r}$.

Set $u = \mu^2(x_N - z)$ and $v = x_N - 2x + z$. Then $\|u\| \leq \mu^2(r+\varepsilon)$ and $\|v\| \leq \mu(r+\varepsilon)$. A direct computation gives

$$\begin{aligned}\|u + v\| &= \|(\mu^2+1)(x_N - x) - (\mu^2-1)(z - x)\| \\ &\geq (\mu^2+1)\Big\| x_N - \Big(\frac{2}{\mu^2+1}x + \frac{\mu^2-1}{\mu^2+1}z\Big)\Big\| \\ &\geq (\mu^2+1)(r-\varepsilon),\end{aligned}$$

and

$$\begin{aligned}\|u - v\| &= \|(\mu^2-1)(x_N - x) - (\mu^2+1)(z - x)\| \\ &\geq (\mu^2+1)\|z - x\| \,\frac{r-\varepsilon}{r}.\end{aligned}$$

Since $\|z - x\| \leq r$, the last quantity is at most $(\mu^2+1)(r-\varepsilon)$. Because λ is non-decreasing and positively homogeneous of degree 1,

$$\begin{aligned}C_{\lambda,t}(X) &\geq \frac{\lambda(\|u+v\|, \|u-v\|)}{(\|u\|^t + \|v\|^t)^{1/t}} \\ &\geq \frac{(\mu^2+1)(r-\varepsilon)}{\mu(r+\varepsilon)(\mu^t+1)^{1/t}}\,\lambda\Big(1, \frac{\|z-x\|}{r}\Big).\end{aligned}$$

Letting $\varepsilon \to 0^+$ yields

$$\lambda\Big(1, \frac{\|z-x\|}{r}\Big) \leq \frac{\mu(\mu^t+1)^{1/t}}{\mu^2+1}\,C_{\lambda,t}(X).$$

Since $z \in A$ was arbitrary, taking the supremum over A gives

$$\lambda\Big(1, \frac{r_C(A)}{r}\Big) \leq \frac{\mu(\mu^t+1)^{1/t}}{\mu^2+1}\,C_{\lambda,t}(X),$$

which completes the proof. □

Corollary 4.1. *Let C be a nonempty bounded closed convex subset of a Banach space X and λ be non-decreasing in both variable such that $C_{\lambda,t}(X) < \frac{\mu^2+1}{\mu(\mu^t+1)^{\frac{1}{t}}}$ and let $T : C \to KC(C)$ be a multivalued nonexpansive mapping. Then T has a fixed point.*

Proof. If $C_{\lambda,t}(X) < \frac{\mu^2+1}{\mu(\mu^t+1)^{\frac{1}{t}}}$, then by Theorem 4.1, $\lambda\Big(1, \frac{r_C(A)}{r}\Big) < 1$. Since λ is a non-decreasing in both variable, $\frac{r_C(A)}{r} < 1$, which means that $r_C(A) < r$. Then, X satisfies the (DL)-condition, then T has a fixed point. □

5. **Results and Discussion**

Remark 5.1. *To illustrate the applicability of our main results, we compute the threshold for $C_{\lambda,t}(X)$ in the classical sequence space l^p $(p \geq 2)$. It is known that for l^p:*

$$R(1, l^p) = 2^{1/p}, \quad \mu(l^p) = 2^{1/p-1}.$$

Substituting these values into Theorem 3.1 (Corollary 3.1), we find that if

$$C_{\lambda,t}(l^p) < \frac{1}{2^{1/t}} \cdot \frac{1}{1 + 2^{-1/p}},$$

then l^p satisfies the (DL)-condition. For instance, when $p = 2$ (Hilbert space), this condition simplifies significantly, confirming that the constant captures the expected geometric rigidity.

This paper establishes a direct link between the generalized geometric constant $C_{\lambda,t}(X)$ and the fixed point property (FPP) for multivalued nonexpansive mappings.

Our main results, Theorems 3.1 and 4.1, demonstrate that $C_{\lambda,t}(X)$ unifies two classical geometric coefficients: the asymptotic coefficient $R(1, X)$ and the weak orthogonality coefficient $\mu(X)$. More precisely, we obtain the following sharp inequalities:

$$r_C\big(A(C, \{x_n\})\big) \leq \frac{2^{1/t}\, R(1, X)}{R(1, X) + 1}\, C_{\lambda,t}(X)\; r(C, \{x_n\}),$$

and

$$\lambda\left(1, \frac{r_C(A)}{r}\right) \leq \frac{\mu(\mu^t + 1)^{1/t}}{\mu^2 + 1}\, C_{\lambda,t}(X).$$

These estimates provide new and computable sufficient conditions for the (DL)-condition to hold.

To place our findings in context, we compare them with two representative lines of research in the literature.

First, Benavides [2] showed that the condition $R(1, X) < 2$ guarantees the FPP. This criterion is simple to check, but it is relatively coarse because it does not take into account the weak orthogonality structure of the space.

Second, Kaewkhao [15] proved that $\mu(X) < 1$ implies the FPP. Although this result is sharp in certain classes of spaces, it imposes a strong requirement on the geometry of X and fails in many reflexive spaces.

Our results overcome these limitations. Indeed, the constant $C_{\lambda,t}(X)$ simultaneously generalizes both $R(1,X)$ and $\mu(X)$, and the thresholds

$$C_{\lambda,t}(X) < \frac{1}{2^{1/t}(1+1/R(1,X))} \quad \text{and} \quad C_{\lambda,t}(X) < \frac{\mu^2+1}{\mu(\mu^t+1)^{1/t}}$$

are strictly stronger than the classical ones. In particular, Corollary 3.1 and Corollary 4.1 show that a sufficiently small $C_{\lambda,t}(X)$ guarantees weak normal structure without requiring $\mu(X) < 1$ or $R(1,X) < 2$ separately.

Consequently, we not only unify several known geometric criteria but also position $C_{\lambda,t}(X)$ as a useful tool for characterizing the geometric structure of Banach spaces.

6. Conclusions

We establishes that the generalized constant $C_{\lambda,t}(X)$ serves as a unified parameter governing the geometric structure of Banach spaces. By bridging $C_{\lambda,t}(X)$ with key coefficients such as $R(1,X)$ and $\mu(X)$, we derive sufficient conditions for the Dominguez-Lorenzo condition and the FPP for multivalued nonexpansive mappings. These results provide a framework for characterizing normal structure and ensuring the existence of fixed points in geometric settings.

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