DISCRETE CHARACTERIZATIONS FOR THE INDIVIDUAL EXPONENTIAL (IN)STABILITY OF EVOLUTIONARY PROCESSES

Yuanyuan WU[1], Tian YUE*[2], Jie ZHANG[3]

In this paper, inspired by the published work (Acta Math. Hungar., 153(1)(2017): pp. 16-23), we further investigate the individual stability and instability of evolutionary processes in Banach spaces. From a discrete-time perspective, we derive new Datko-type criteria for individual exponential stability. Furthermore, analogous characterizations for individual exponential instability are also presented.

Keywords: individual exponential (in)stability, evolutionary process, Datko-type results
MSC2020: 34D05, 34D09

1. Introduction

As is well known, the concept of exponential stability is pivotal in the qualitative theory of dynamical systems. It plays a crucial role in the characterizations and applications of dichotomy, admissibility, spectral theory, robustness, invariant manifolds and linearization theory. It is worth mentioning that Datko [4] achieved a particularly significant result in the stability theory of evolutionary processes in 1972, which is detailed as follows.

Theorem 1.1. *Let $\Phi = \{\Phi(t, t_0)\}_{t \geq t_0 \geq 0}$ be an evolutionary process of class $C(0, e)$ on a Banach space X. Then it is uniformly exponentially stable if and*

[1]Student Affairs Office, Hubei University of Automotive Technology, Shiyan 442002, China

[2]*Corresponding author. Department of Mathematics, Hubei University of Automotive Technology, Shiyan 442002, China, e-mail: yuet@huat.edu.cn

[3]Department of Mathematics, Hubei University of Automotive Technology, Shiyan 442002, China

only if there exists $p \geq 1$ *such that*

$$\sup_{t_0 \geq 0} \int_{t_0}^{\infty} \|\Phi(\tau, t_0)x\|^p \, d\tau < \infty, \ \forall x \in X.$$

Datko's result has inspired a considerable number of recent papers on exponential stability of evolution equations in Banach spaces. Here we mention the studies [5, 6, 8, 12, 14, 17, 18, 19] dealing with the characterizations of (non)uniform exponential stability for integral equations and the works [7, 9, 11] considering the case of difference equations.

Currently, there is a vast literature on the classical theory and applications of exponential stability, see e.g. monographs by Sell and You [16] and by Barreira and Valls [1]. However, when it comes to dynamical behavior (especially in non-autonomous case), the classical (uniform) concept of exponential stability is extremely restrictive since it requires that the contraction rate be uniformly exponential for all points. Therefore, it is very important to explore more general types of exponential behavior. This motivates the introduction of the concept of individual exponential behavior. The notion of individual asymptotic behaviour originates from the landmark work of Chicone and Latushkin [3]. Afterwards, the discussion of individual exponential stability of dynamical systems using different approaches attracted the attention of several researchers. For instance, Buşe and Pogan [2] investigated the individual exponential stability of evolution families by using normed function spaces. Preda, Pogan and Preda [15] studied the exponential stability of certain individual orbits of an evolutionary process based on the individual admissibility of some pairs of Schäffer spaces. In [10], Mureşan, Preda and Preda used the Datko's approach to establish integral characterizations both for individual exponential stability and instability of evolutionary processes. More precisely, they proved that Datko's result remains valid if the statement is required only for one vector x (instead of globally). In addition, Preda, Preda and Onofrei [13] brought an innovation by choosing independently the constant in the boundedness theorem for each value of the metric space variable. Using the standard Perron's method, they investigated the individual exponential stability of linear skew -product semiflows over semiflows.

Inspired by [10], this paper establishes the sufficient conditions for individual exponential stability and instability of evolution processes on Banach spaces based on its foundation. Specifically, from a discrete-time viewpoint, we derive several sufficient conditions for the individual exponential stability and instability by leveraging the results of [10] in continuous-time.

2. **Preliminaries**

In this section, we will present some notations, definitions, and technical theorems that will be utilized in the subsequent content. Let X be a real or complex Banach space and $\mathcal{B}(X)$ the set of all bounded linear operators acting on X. The norms on X and $\mathcal{B}(X)$ will be denoted by $\|\cdot\|$. We also denote by $\mathbf{R}$ the set of real numbers, by $\mathbf{R}_+ = [0, +\infty)$ and by $\Delta = \left\{(t,s) \in \mathbf{R}_+^2 : t \geq s\right\}$. For any given real number t, the symbol $[t]$ represents the greatest integer that is less than or equal to t.

Definition 2.1. (see [4]) *We say that $\Phi = \{\Phi(t,t_0)\}_{t\geq t_0 \geq 0} \subset \mathcal{B}(X)$ is an evolutionary process if the following properties are satisfied:*

(i) $\Phi(t,t) = Id$ (the identity operator on X), for all $t \in \mathbf{R}_+$;

(ii) $\Phi(t,\tau)\Phi(\tau,t_0) = \Phi(t,t_0)$, for all $t \geq \tau \geq t_0 \geq 0$;

(iii) $\Phi(\cdot,t_0)x$ is continuous on $[t_0,\infty)$, for all $(t_0,x) \in \mathbf{R}_+ \times X$;

$\Phi(t,\cdot)x$ is continuous on $[0,t]$, for all $(t,x) \in \mathbf{R}_+ \times X$.

In addition, if there exist $M, \omega > 0$ such that

$$\|\Phi(t,t_0)\| \leq M e^{\omega(t-t_0)} \tag{1}$$

for all $(t,t_0) \in \Delta$, then we say that Φ is an evolutionary process with uniform exponential growth, or we say that Φ is an evolutionary process of class $C(0,e)$.

Definition 2.2. (see [4]) *An evolutionary process $\Phi = \{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ is said to be:*

(i) uniformly exponentially stable if there exist $N, v > 0$ such that

$$\|\Phi(t,t_0)x\| \leq N e^{-v(t-t_0)} \|x\|, \ \forall (t,t_0,x) \in \Delta \times X;$$

(ii) uniformly exponentially unstable if there exist $N, v > 0$ such that

$$\|\Phi(t,t_0)x\| \geq N e^{v(t-t_0)} \|x\|, \ \forall (t,t_0,x) \in \Delta \times X.$$

Remark 2.1. *An evolutionary process $\Phi = \{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ is uniformly exponentially stable if and only if there are $N, v > 0$ such that*

$$\|\Phi(t,t_0)x\| \leq N e^{-v(t-s)} \|\Phi(s,t_0)x\|,$$

for all $t \geq s \geq t_0 \geq 0$ and $x \in X$.

This implies that the trajectory $\Phi(\cdot, t_0)x$ is uniformly exponentially stable, for all initial values $(t_0, x) \in \mathbf{R}_+ \times X$.

Definition 2.3. (see [10]) *An evolutionary process* $\Phi = \{\Phi(t, t_0)\}_{t \geq t_0 \geq 0}$ *is said to be:*

(i) individual exponentially stable if for a fixed pair $(t_0, x) \in \mathbf{R}_+ \times X$ *there exist* $N(t_0), v(t_0) > 0$ *such that*

$$\|\Phi(t, t_0)x\| \leq N(t_0)e^{-v(t_0)(t-t_0)} \|x\|, \ \forall t \geq t_0;$$

(ii) individual exponentially unstable if for a fixed pair $(t_0, x) \in \mathbf{R}_+ \times X$ *there exist* $N(t_0), v(t_0) > 0$ *such that*

$$\|\Phi(t, t_0)x\| \geq N(t_0)e^{v(t_0)(t-t_0)} \|x\|, \ \forall t \geq t_0.$$

Remark 2.2. *If* Φ *is uniformly exponentially stable, then it is individual exponentially stable. The converse does not necessarily hold (see* [10]*, Example 4.1). Similarly, if* Φ *is individual exponentially unstable, this does not necessarily imply that* Φ *is uniformly exponentially unstable. To show this, we consider the following example.*

Example 2.1. *Let* $X = \mathbf{R}$*. An evolutionary process defined by*

$$\Phi(t, t_0)x = e^{8t-3tsint-8t_0+3t_0sint_0}x, \ \forall (t, t_0, x) \in \Delta \times X.$$

For a fixed pair $(t_0, x) \in \mathbf{R}_+ \times X$*, we have*

$$\begin{aligned} |\Phi(t, t_0)x| &= e^{8t-3t+3t(1-sint)-8t_0+3t_0-3t_0(1-sint_0)} |x| \\ &= \frac{e^{3t(1-sint)}}{e^{3t_0(1-sint_0)}} e^{5(t-t_0)} |x| \\ &\geq e^{-6t_0} e^{5(t-t_0)} |x|, \end{aligned}$$

for all $t \geq t_0$*. Thus Definition 2.3 (ii) is satisfied for* $N(t_0) = e^{-6t_0}$ *and* $v(t_0) = 5$.

If we suppose that Φ *is uniformly exponentially unstable, then there exist* $N, v > 0$ *such that*

$$e^{8t-3tsint-8t_0+3t_0sint_0} \geq Ne^{v(t-t_0)}$$

for all $(t, t_0) \in \Delta$*. In particular, for* $t = 2k\pi + \frac{\pi}{2}$ *and* $t_0 = 2k\pi$*, we obtain that*

$$Ne^{6k\pi + \frac{v\pi}{2}} \leq e^{\frac{5\pi}{2}},$$

which is false for $k \to +\infty$*. Hence,* Φ *is not uniformly exponentially unstable.*

Theorem 2.1. (see [10]) *Let* $\Phi = \{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ *be an evolutionary process of class* $C(0,e)$. *If for a fixed pair* $(t_0,x) \in \mathbf{R}_+ \times X$ *there exist* $k(t_0), p > 0$ *such that*

$$\left(\int_t^\infty \|\Phi(\tau,t_0)x\|^p \, d\tau\right)^{1/p} \leq k(t_0) \, \|\Phi(t,t_0)x\| \,, \;\; \forall t \geq t_0, \tag{2}$$

then Φ *is individual exponentially stable.*

Theorem 2.2. (see [10]) *Let* $\Phi = \{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ *be an evolutionary process of class* $C(0,e)$. *If for fixed* $x \in X\setminus\{0\}$ *and* $t_0 \geq 0$ *with* $\Phi(t,t_0)x \neq 0$ $(\forall t \geq t_0)$ *there exist* $k(t_0), p > 0$ *such that*

$$\left(\int_t^\infty \frac{d\tau}{\|\Phi(\tau,t_0)x\|^p}\right)^{1/p} \leq \frac{k(t_0)}{\|\Phi(t,t_0)x\|}, \;\; \forall t \geq t_0, \tag{3}$$

then Φ *is individual exponentially unstable.*

3. **Main results**

The analysis presented in this section aims to establish discrete characterizations pertaining to the individual exponential stability and instability of evolutionary processes. In this section, we assume that Φ is an evolutionary process of class $C(0,e)$.

To begin with, we will introduce two Datko-type characterizations for the notion of individual exponential stability.

Theorem 3.1. *If for a fixed pair* $(t_0,x) \in \mathbf{R}_+ \times X$ *there exist* $\vartheta(t_0), p > 0$ *such that*

$$\left(\sum_{n=0}^{\infty} \|\Phi(t+n,t_0)x\|^p\right)^{1/p} \leq \vartheta(t_0) \, \|\Phi(t,t_0)x\| \,, \;\; \forall t \geq t_0, \tag{4}$$

then Φ *is individual exponentially stable.*

Proof. Let $x \in X$ and $t_0 \geq 0$. Using (1) and (4), we have

$$\begin{aligned}\int_t^\infty \|\Phi(\tau,t_0)x\|^p \, d\tau &= \sum_{n=0}^{\infty} \int_{t+n}^{t+n+1} \|\Phi(\tau,t_0)x\|^p \, d\tau \\ &= \sum_{n=0}^{\infty} \int_{t+n}^{t+n+1} \|\Phi(\tau,t+n)\Phi(t+n,t_0)x\|^p \, d\tau\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{n=0}^{\infty} \int_{t+n}^{t+n+1} (Me^{\omega})^p \left\| \Phi(t+n, t_0)x \right\|^p d\tau \\
&= (Me^{\omega})^p \sum_{n=0}^{\infty} \left\| \Phi(t+n, t_0)x \right\|^p \\
&\leq (Me^{\omega})^p \vartheta^p(t_0) \left\| \Phi(t, t_0)x \right\|^p .
\end{aligned}$$

Hence, the relation (2) holds with $k(t_0) = Me^{\omega}\vartheta(t_0)$. Now by Theorem 2.1, we conclude that Φ is individual exponentially stable. □

Theorem 3.2. *If for a fixed pair* $(t_0, x) \in \mathbf{R}_+ \times X$ *there exist* $\vartheta(t_0), p > 0$ *such that*

$$\left(\sum_{n=[t]+1}^{\infty} \left\| \Phi(n, t_0)x \right\|^p \right)^{1/p} \leq \vartheta(t_0) \left\| \Phi(t, t_0)x \right\|, \ \forall t \geq t_0, \tag{5}$$

then Φ *is individual exponentially stable.*

Proof. Let $x \in X$ and $t_0 \geq 0$. It follows from (1) and (5) that

$$\begin{aligned}
&\int_t^{\infty} \left\| \Phi(\tau, t_0)x \right\|^p d\tau \\
&\quad = \int_t^{[t]+1} \left\| \Phi(\tau, t_0)x \right\|^p d\tau + \int_{[t]+1}^{\infty} \left\| \Phi(\tau, t_0)x \right\|^p d\tau \\
&\quad = \int_t^{[t]+1} \left\| \Phi(\tau, t)\Phi(t, t_0)x \right\|^p d\tau + \sum_{n=[t]+1}^{\infty} \int_n^{n+1} \left\| \Phi(\tau, n)\Phi(n, t_0)x \right\|^p d\tau \\
&\quad \leq (Me^{\omega})^p \int_t^{[t]+1} \left\| \Phi(t, t_0)x \right\|^p d\tau + \sum_{n=[t]+1}^{\infty} \int_n^{n+1} (Me^{\omega})^p \left\| \Phi(n, t_0)x \right\|^p d\tau \\
&\quad \leq (Me^{\omega})^p \left\| \Phi(t, t_0)x \right\|^p + (Me^{\omega})^p \sum_{n=[t]+1}^{\infty} \left\| \Phi(n, t_0)x \right\|^p \\
&\quad \leq (Me^{\omega})^p (1 + \vartheta^p(t_0)) \left\| \Phi(t, t_0)x \right\|^p ,
\end{aligned}$$

which immediately implies that (2) holds with $k(t_0) = Me^{\omega}(1 + \vartheta^p(t_0)^{1/p}$. Therefore, by Theorem 2.1, we conclude that Φ is individual exponentially stable. □

Next, we extend the techniques employed in Theorem 3.2 to the case of a individual exponential instability.

Theorem 3.3. *If for fixed $x \in X \setminus \{0\}$ and $t_0 \geq 0$ with $\Phi(t,t_0)x \neq 0$ $(\forall t \geq t_0)$ there exist $\vartheta(t_0), p > 0$ such that*

$$\left(\sum_{n=[t]+1}^{\infty} \frac{1}{\|\Phi(n,t_0)x\|^p} \right)^{1/p} \leq \frac{\vartheta(t_0)}{\|\Phi(t,t_0)x\|}, \ \forall t \geq t_0, \tag{6}$$

then Φ is individual exponentially unstable.

Proof. Let $x \in X \setminus \{0\}$, $t_0 \geq 0$ and $t \geq t_0$. If $t \leq \tau < [t]+1$, then by (1), we have

$$\|\Phi([t]+1,t_0)x\| = \|\Phi([t]+1,\tau)\Phi(\tau,t_0)x\| \leq (Me^{\omega}) \|\Phi(\tau,t_0)x\|. \tag{7}$$

From (6) and (7) we obtain that

$$\frac{1}{\|\Phi(\tau,t_0)x\|} \leq \frac{Me^{\omega}}{\|\Phi([t]+1,t_0)x\|} \leq \frac{Me^{\omega}\vartheta(t_0)}{\|\Phi(t,t_0)x\|}.$$

Consequently,

$$\int_t^{[t]+1} \frac{d\tau}{\|\Phi(\tau,t_0)x\|^p} \leq \frac{(Me^{\omega}\vartheta(t_0))^p}{\|\Phi(t,t_0)x\|^p}. \tag{8}$$

If $\tau \geq [t]+1$, we denote $m = [\tau]$. Then, it follows from (1) that

$$\|\Phi(m+1,t_0)x\| = \|\Phi(m+1,\tau)\Phi(\tau,t_0)x\| \leq (Me^{\omega}) \|\Phi(\tau,t_0)x\|,$$

and thus

$$\int_m^{m+1} \frac{d\tau}{\|\Phi(\tau,t_0)x\|^p} \leq \frac{(Me^{\omega})^p}{\|\Phi(m+1,t_0)x\|^p}. \tag{9}$$

From (6) and (9) we obtain that

$$\begin{aligned} \int_{[t]+1}^{\infty} \frac{d\tau}{\|\Phi(\tau,t_0)x\|^p} &= \sum_{m=[t]+1}^{\infty} \int_m^{m+1} \frac{d\tau}{\|\Phi(\tau,t_0)x\|^p} \\ &\leq (Me^{\omega})^p \sum_{m=[t]+1}^{\infty} \frac{1}{\|\Phi(m+1,t_0)x\|^p} \\ &= (Me^{\omega})^p \sum_{i=[t]+2}^{\infty} \frac{1}{\|\Phi(i,t_0)x\|^p} \\ &\leq \frac{(Me^{\omega}\vartheta(t_0))^p}{\|\Phi(t,t_0)x\|^p}. \end{aligned} \tag{10}$$

Combining (8) with (10), we have that

$$\int_t^{\infty} \frac{d\tau}{\|\Phi(\tau,t_0)x\|^p} = \int_t^{[t]+1} \frac{d\tau}{\|\Phi(\tau,t_0)x\|^p} + \int_{[t]+1}^{\infty} \frac{d\tau}{\|\Phi(\tau,t_0)x\|^p} \leq \frac{2(Me^{\omega}\vartheta(t_0))^p}{\|\Phi(t,t_0)x\|^p}.$$

Hence, the relation (3) holds with $k(t_0) = 2^{1/p} M e^{\omega} \vartheta(t_0)$. By virtue of Theorem 2.2, we conclude that Φ is individual exponentially unstable. $\square$

Finally, by appending an exponential decay condition to Φ, we will present another Datko-type characterization of individual exponential instability.

Theorem 3.4. *Let $\Phi = \{\Phi(t,t_0)\}_{t \geq t_0 \geq 0}$ be an evolutionary process with uniform exponential decay, that is, there exist $K, \alpha > 0$ such that*

$$\|\Phi(t,t_0)x\| \geq K e^{-\alpha(t-t_0)} \|x\|, \ \forall (t,t_0,x) \in \Delta \times X. \tag{11}$$

If for fixed $x \in X \setminus \{0\}$ and $t_0 \geq 0$ with $\Phi(t,t_0)x \neq 0$ ($\forall t \geq t_0$) there exist $\vartheta(t_0), p > 0$ such that

$$\left(\sum_{n=0}^{\infty} \frac{1}{\|\Phi(t+n,t_0)x\|^p} \right)^{1/p} \leq \frac{\vartheta(t_0)}{\|\Phi(t,t_0)x\|}, \ \forall t \geq t_0, \tag{12}$$

then Φ is individual exponentially unstable.

Proof. Let $x \in X \setminus \{0\}$, $t_0 \geq 0$ and $t \geq t_0$. Using (11), for every natural number $n \geq 0$ and every $\tau \in [t+n, t+n+1)$ we have

$$\|\Phi(\tau,t_0)x\| = \|\Phi(\tau,t+n)\Phi(t+n,t_0)x\| \geq K e^{-\alpha(\tau-t-n)} \|\Phi(t+n,t_0)x\|,$$

and thus

$$\frac{1}{\|\Phi(\tau,t_0)x\|^p} \leq \frac{K^{-p} e^{\alpha p(\tau-t-n)}}{\|\Phi(t+n,t_0)x\|^p}. \tag{13}$$

By (12) and (13), we have

$$\begin{aligned}
\int_t^{\infty} \frac{d\tau}{\|\Phi(\tau,t_0)x\|^p} &= \sum_{n=0}^{\infty} \int_{t+n}^{t+n+1} \frac{d\tau}{\|\Phi(\tau,t_0)x\|^p} \\
&\leq \sum_{n=0}^{\infty} \int_{t+n}^{t+n+1} \frac{K^{-p} e^{\alpha p(\tau-t-n)}}{\|\Phi(t+n,t_0)x\|^p} d\tau \\
&\leq \sum_{n=0}^{\infty} \frac{K^{-p} e^{\alpha p}}{\|\Phi(t+n,t_0)x\|^p} \\
&\leq \frac{K^{-p} e^{\alpha p} \vartheta^p(t_0)}{\|\Phi(t,t_0)x\|^p}.
\end{aligned}$$

Hence, the relation (3) holds with $k(t_0) = K^{-1} e^{\alpha} \vartheta(t_0)$. Now by Theorem 2.2, we conclude that Φ is individual exponentially unstable. $\square$

4. Conclusions

In this paper, we have studied the discrete-time characterizations of individual (in)stability for evolutionary processes in Banach spaces. More precisely, under the assumption of uniform exponential growth, we established two Datko-type theorems on individual exponential stability by using a discrete-time approach (see Theorems 3.1 and 3.2). In parallel, similar characterizations of individual exponential instability are also presented (see Theorems 3.3 and 3.4). In the future, we will extend the results of this paper to the stability analysis of non-autonomous systems arising in population dynamics and epidemiological modeling.

Acknowledgments

The authors are indebted to the reviewers and editors for their careful reading and constructive comments, which greatly improved the contents and exposition of this paper.

REFERENCES

[1] *L. Barreira and C. Valls*, Stability of Nonautonomous Differential Equations, Lecture Notes in Math., Vol. 1926, Springer, Berlin, 2008.

[2] *C. Buşe and A. Pogan*, Individual exponential stability for evolution families of linear and bounded operators, New Zealand J. Math., **30**(2001), No. 1, 15–24.

[3] *C. Chicone and Y. Latushkin*, Evolution Semigroups in Dynamical Systems and Differential Equations, Mathematical Surveys and Monographs, Vol. 70, American Mathematical Society, Providence, 1999.

[4] *R. Datko*, Uniform asymptotic stability of evolutionary processes in a Banach space, SIAM J. Math. Anal., **3**(1972), No. 3, 428–445.

[5] *D. Dragičević*, Datko-Pazy conditions for nonuniform exponential stability, J. Difference Equ. Appl., **24**(2018), No. 3, 344–357.

[6] *D. Dragičević*, Strong nonuniform behaviour: a Datko type characterization, J. Math. Anal. Appl., **459**(2018), No. 1, 266–290.

[7] *D. Dragičević, A. L. Sasu, B. Sasu and A. Şirianţu*, Nonuniform criteria for exponential stability of discrete and continuous variational dynamics, J. Math. Anal. Appl., **554**(2026), No. 2, Art. ID 129964.

[8] *P. V. Hai*, Continuous and discrete characterizations for the uniform exponential stability of linear skew-evolution semiflows, Nonlinear Anal., **72**(2010), No. 12, 4390–4396.

[9] *N. Lupa and L. H. Popescu*, A complete characterization of exponential stability for discrete dynamics, J. Difference Equ. Appl., **23**(2017), No. 12, 2072–2092.

[10] *M. Mureşan, C. Preda and P. Preda*, Individual stability and instability for evolutionary processes, Acta Math. Hungar., **153**(2017), No. 1, 16–23.

[11] *I.-L. Popa, T. Ceauşu and M. Megan*, On exponential stability for linear discrete-time systems in Banach spaces, Comput. Math. Appl., **63**(2012), No. 11, 1497–1503.

[12] *C. Preda, P. Preda and F. Bătăran*, An extension of a theorem of R. Datko to the case of (non)uniform exponential stability of linear skew-product semiflows, J. Math. Anal. Appl., **425**(2015), No. 2, 1148–1154.

[13] *C. Preda, P. Preda and O. R. Onofrei*, Individual exponential stability for linear skew-products semiflows over a semiflow, Period. Math. Hungar., **79**(2019), No. 2, 168–176.

[14] *P. Preda and M. Megan*, Exponential dichotomy of evolutionary processes in Banach spaces, Czechoslovak Math. J., **35**(1985), No. 2, 312–323.

[15] *P. Preda, A. Pogan and C. Preda*, Individual stability for evolutionary processes, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., **13**(2006), No. 5, 525–536.

[16] *G. Sell and Y. You*, Dynamics of Evolutionary Equations, Appl. Math. Sci., Vol. 143, Springer, New York, 2002.

[17] *C. Stoica and M. Megan*, On uniform exponential stability for skew-evolution semiflows on Banach spaces, Nonlinear Anal., **72**(2010), No. 3–4, 1305–1313.

[18] *T. M. Személy Fülöp*, Datko type characterizations for uniform dichotomy in mean with growth rates for reversible stochastic skew-evolution semiflows in Banach spaces, Glas. Mat. Ser. III, **60**(2025), No. 1, 167–182.

[19] *N. Van Minh, F. Räbiger and R. Schnaubelt*, Exponential stability, exponential expansiveness, and exponential dichotomy of evolution equations on the half-line, Integral Equ. Oper. Theory, **32**(1998), No. 3, 332–353.