MIZOGUCHI AND TAKAHASHI'S θ-TYPE FIXED POINT THEOREM IN v-GENERALIZED METRIC SPACES

Wei MING[1], Tayyab KAMRAN[*2], Hamza KHAN[3], Umar ISHTIQ[4], Mohammad AKRAM[*5]

In this paper we introduce the notion of Mizoguchi-Takahashi's θ-type contraction and obtain a fixed point theorem. We have established nontrivial examples to show that our result not only generalizes Nadler's fixed point theorem in v-generalized metric spaces but it also generalizes Nadler's fixed point theorem, Mizoguchi and Takahashi fixed point theorem and some results by Feng-Liu, Klim and Wardowski, Pathak and Shahzad, and Kamran and Kiran.

Keywords: v-Generalized metric, v-Cauchy sequence, v-complete metric space, $(\sum, \neq)$-Cauchy sequence, $(\sum, \neq)$ T-orbitally Cauchy sequence, Mizoguchi-Takahashi's θ-type contraction

MSC2010: 47H10, 54H25

1. Introduction

Banach-Caccioppoli's fixed point theorem [1, 2] is a fundamental tool for nonlinear problems. Nadler [3] extended Banach-Caccioppoli's fixed point theorem to multivalued maps. Branciari [4] generalized the notion of metric space by replacing the triangle inequality with a quadrilateral inequality and called such space as generalized metric space. Branciari extended Banach-Caccioppoli's fixed point theorem to generalized metric spaces. It was pointed out that the notion of generalized metric as defined by Branciari lacks some basic topological properties and hence the generalizations of Banach fixed point

[1]School of Mathematics and Statistics, Hubei Normal University, Huangshi-435002, China, e-mail: weiweidashan2025@163.com

[2]Department of Mathematics, Quaid-i-Azam University, Islamabad-45320, Pakistan, Jadara Research Center, Jadara University, Irbid, Jordan, Center for Theoretical Physics, Khazar University, 41 Mehseti Str., Baku, AZ1096, Azerbaijan, e-mail: tayyabkamran@gmail.com, tkamran@qau.edu.pk

[3]Department of Mathematics, Quaid-i-Azam University, Islamabad-45320, Pakistan, e-mail: hamza.22311016@math.qau.edu.pk

[4]Office of Research, Innovation and Commercialization, University of Management and Technology, Lahore-54770, Pakistan, email: umarishtiaq@umt.edu.pk

[5]Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah 42351, Saudi Arabia, email: akramkhan_20@rediffmail.com

theorem, in the setting of the generalized metric space as defined by Branciari, are not completely correct. To overcome these difficulties, many authors imposed additional conditions, usually Hausdorffness of the induced topology. Jleli and Samet [5], Samet [6] and Turinici [7] showed that such hypotheses can often be replaced by mild adjustments in the statements, making most results valid without extra topological assumptions.

Motivated by these developments, Suzuki and his coauthors [16, 17] further generalized the notion of generalized metric and introduced v-generalized metric spaces. Moreover, they proved Nadler's fixed point theorem in v-generalized metric spaces.

The Banach-Caccioppoli [1, 2] fixed point theorem states that every contraction mapping on a complete metric space has a distinct fixed point, and that the iterates of any point under the contraction mapping converge to the fixed point. Banach obtained this result for complete normed spaces [1] by using successive approximations, a method used by Cauchy, Liouville, Lipschitz, Peano and Picard in their work. Caccioppoli extended Banach's theorem to complete metric spaces [2]. The Banach-Caccioppoli's principle is a fundamental tool for nonlinear problems; see Brooks and Schmitt [8]. Following commemorate theorem explicitly appeared in 1930.

Theorem 1.1 ([1, 2]). *Let $(\mathcal{X}, d)$ be a complete metric space and f be a contraction on $\mathcal{X}$, i.e., there exist a real number $0 \leq \lambda < 1$ such that for all $p_1, q_1 \in \mathcal{X}$,*

$$d(fp_1, fq_1) \leq \lambda d(p_1, q_1).$$

Then f has a unique fixed point.

Nadler [3] initiated the study of fixed points for multivalued mappings. The scope of economic, sociology and biology in macrosystems is a multifaceted phenomena. Differential inclusions serve as natural models in macrosystems with hysteresis [9], and fixed point theorems for multivalued maps play a crucial role in solving them.

Nadler [3] made use of the notion of Hausdorff metric on the set of all nonempty, closed and bounded subsets of a metric space to establish fixed point theorem for multivalued mappings. Now we recall the definition of the Hausdorff metric, for ready reference. Let $\mathcal{N}(\mathcal{X})$ be the family of all nonempty subsets of $\mathcal{X}$, and denote by $\mathcal{B}(\mathcal{X}), \mathcal{CL}(\mathcal{X}), \mathcal{CB}(\mathcal{X})$ and $\mathcal{K}(\mathcal{X})$ the families of nonempty bounded, closed, closed and bounded, and compact subsets of a metric space $\mathcal{X}$, respectively.

Definition 1.1. *For $A, B \in \mathcal{CB}(\mathcal{X})$,*

$$\mathcal{H}(A, B) = \max\{\sup_{p_1 \in A} d(p_1, B), \sup_{q_1 \in B} d(q_1, A)\}, \tag{1}$$

*defines a metric on $\mathcal{CB}(\mathcal{X})$. It is called **Hausdorff metric** on $\mathcal{CB}(\mathcal{X})$. It is well known fact that if $(\mathcal{X}, d)$ is complete then $(\mathcal{CB}(\mathcal{X}), \mathcal{H})$ is also complete.*

In 1969, Nadler [3] extended Banach-Caccioppoli contraction principle (Theorem 1.1) in the following way.

Theorem 1.2 ([3], Theorem 5)**.** *Let $(\mathcal{X}, d)$ be a complete metric space and 0 $\leq \lambda < 1$. Assume that $T : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ be such that*

$$\mathcal{H}(Tp_1, Tq_1) \leq \lambda d(p_1, q_1) \quad \text{for all } p_1, q_1 \in \mathcal{X}.$$

Then T has a fixed point, i.e., there is a point $u \in \mathcal{X}$ such that $u \in Tu$.

Now we present some important generalizations of the Nadler fixed point theorem (Theorem 1.2) existing in the literature.

Reich, in 1972, generalized Nadler's fixed point theorem by using an auxiliary function α in the following way.

Theorem 1.3 ([10])**.** *Let $(\mathcal{X}, d)$ be a complete metric space and $T : \mathcal{X} \to \mathcal{K}(\mathcal{X})$. Assume that there exist a map $\alpha : [0, \infty) \to [0, 1)$ such that*

$$\forall_{t \in [0,\infty)} \big\{ \lim_{r \to t^+} \sup \alpha(r) < 1 \big\}$$

and

$$\forall_{p_1, q_1 \in \mathcal{X},\ p_1 \neq q_1} \ \big\{ \mathcal{H}(Tp_1, Tq_1) \leq \alpha(d(p_1, q_1)) d(p_1, q_1) \big\}.$$

Then T has a fixed point.

To get the extension, Reich's [10] has to restrict the range of the mapping T to $\mathcal{K}(\mathcal{X})$. Reich [10] posed the question that is it possible to replace the range of T by $\mathcal{CB}(\mathcal{X})$ or $\mathcal{CL}(\mathcal{X})$. Mizoguchi and Takahashi provided a positive answer to the question of Reich [10] and proved the following result in 1989.

Theorem 1.4 ([11], Theorem 5)**.** *Let $(\mathcal{X}, d)$ be a complete metric space and $T : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$. Assume that there exist a map $\alpha : [0, \infty) \to [0, 1)$ such that*

$$\forall_{t \in [0,\infty)} \big\{ \lim_{r \to t^+} \sup \alpha(r) < 1 \big\}$$

and

$$\forall_{p_1, q_1 \in \mathcal{X},\ p_1 \neq q_1} \ \big\{ \mathcal{H}(Tp_1, Tq_1) \leq \alpha(d(p_1, q_1)) d(p_1, q_1) \big\}.$$

Then T has a fixed point.

In 2007, Klim and Wardowski [12], inspired by Mizoguchi and Takahashi and using ideas of Feng and Liu [13], obtained the following result.

For $p_1 \in \mathcal{X}$, define

$$I_1^{p_1} = \{q_1 \in Tp_1 : d(p_1, q_1) \leq d(p_1, Tp_1)\}. \tag{2}$$

Theorem 1.5 ([12], Theorem 2.2)**.** *Let $(\mathcal{X}, d)$ be a complete metric space and $T : \mathcal{X} \to \mathcal{K}(\mathcal{X})$. Assume that the following hold:*
(i) *The map $f : \mathcal{X} \to \mathbb{R}$ defined by $f(p_1) = d(p_1, Tp_1)$, $p_1 \in \mathcal{X}$, is lower semi-continuous;*
(ii) *there exist $\alpha : [0, \infty) \to [0, 1)$ such that*

$$\forall_{t \in [0,\infty)} \big\{ \lim_{r \to t^+} \sup \alpha(r) < 1 \big\}$$

and

$$\forall_{p_1\in\mathcal{X}}\exists_{q_1\in I_1^{p_1}}\{d(q_1,Tq_1)\leq \alpha(d(p_1,q_1))d(p_1,q_1)\},$$

where $I_1^{p_1}$ *is defined in (2). Then* T *has a fixed point.*

In 2009, Pathak and Shahzad [14] introduced a new class of mappings and generalized Theorem 1.5 in the following way.

Definition 1.2 ([14], Page 2634)**.** *Let* $\mathbb{R}$ *denotes the set of real numbers. For* $\Lambda\in(0,+\infty]$ *let* $\Theta[0,\Lambda)$ *represent the class of functions* $\theta:[0,\Lambda)\to\mathbb{R}$ *satisfying the following conditions:*

(i) θ *is nondecreasing on* $[0,\Lambda)$;
(ii) $\theta(t)>0$ *for each* $t\in(0,\Lambda)$;
(iii) θ *is subadditive in* $(0,\Lambda)$, *i.e.,* $\theta(t_1+t_2)\leq\theta(t_1)+\theta(t_2)$ *for* $t_1,t_2\in(0,\Lambda)$.

Remark 1.1. *Note that* (i) *implies that* θ *is strictly isotone on* $(0,\Lambda)$; *i.e.,*

$$\theta(t_1)<\theta(t_2)\Rightarrow t_1<t_2,\ t_1,t_2\in(0,\Lambda). \tag{3}$$

Let $(\mathcal{X},d)$ be a metric space and let $D=\sup\{d(x,y):x,y\in\mathcal{X}\}$. Subsequently, throughout this section, we set $\Lambda=D$ if $D=\infty$ and $\Lambda>D$ if $D<\infty$. Define

$$\mathcal{M}(1,p_1;\theta)=\big\{q_1\in Tp_1:\theta(d(p_1,q_1))\leq\theta(d(p_1,Tp_1))\big\}. \tag{4}$$

Theorem 1.6 ([14], Theorem 2.4)**.** *Let* $(\mathcal{X},d)$ *be a complete metric space and* $T:\mathcal{X}\to\mathcal{CL}(\mathcal{X})$. *Assume that the following conditions hold:*
(i) *The map* $f:\mathcal{X}\to\mathbb{R}$ *defined by* $f(p_1)=d(p_1,Tp_1),\ p_1\in\mathcal{X}$, *is lower semicontinuous;*
(ii) *There exists* $\alpha:(0,\infty)\to[0,1)$ *such that*

$$\forall_{t\in[0,\infty)}\big\{\lim_{r\to t^+}\sup\alpha(r)<1\big\}.$$

(iii) *There exists* $\theta\in\Theta[0,\Lambda)$ *satisfying the following condition:*

$$\textit{for all } p_1\in\mathcal{X},\ \mathcal{M}(1,p_1;\theta)\ \textit{is nonempty}$$

and

$$\forall p_1\in\mathcal{X}\ \exists q_1\in\mathcal{M}(1,p_1;\theta)\ \big\{\theta(d(q_1,Tq_1))\leq\alpha(d(p_1,q_1))\theta(d(p_1,q_1))\big\},$$

where $\mathcal{M}(1,p_1;\theta)$ *is defined in (4). Then* T *has a fixed point.*

In 2011, Kamran and Kiran [15] introduced a new class of functions which is a subclass of $\Theta[0,\Lambda)$ in the following manner.

Definition 1.3. *Let* $\Theta_h[0,\Lambda)$ *denote the subclass of* $\Theta[0,\Lambda)$ *consisting of those functions from* $\Theta[0,\Lambda)$ *which are positive homogeneous. That is,*

$$\theta(at)\leq a\theta(t)\ \forall\ a>0,\ t\in[0,\Lambda). \tag{5}$$

Definition 1.4 ([15])**.**

(i) *Let* T *be a mapping from* $\mathcal{X}$ *into* $\mathcal{CL}(\mathcal{X})$ *and* $p_0\in\mathcal{X}$, *then* $O(p_0,T)=\{p_0,p_1,\ p_2,\cdots\}$, *where* $p_n\in Tp_{n-1}$, *is called* ***orbit of*** T *at* p_0.

(ii) *A mapping $f : \mathcal{X} \to \mathbb{R}$ is said to be T-**orbitally lower semi continuous** if $\{p_n\}$ is a sequence in $O(p_0, T)$ and $p_n \to \xi$ implies $f(\xi) \leq \lim_n \inf f(p_n)$.*

Theorem 1.7 ([15], Theorem 4.2)**.** *Let $(\mathcal{X}, d)$ be a complete metric space and $T : \mathcal{X} \to \mathcal{CL}(\mathcal{X})$ satisfying*

$$\theta(d(q_1, Tq_1)) \leq \alpha(d(p_1, q_1))\theta(d(p_1, q_1)) \quad \text{for all } p_1 \in \mathcal{X} \text{ and } q_1 \in Tp_1,$$

where α is a function from $(0, \infty)$ into $[0, 1)$ such that

$$\limsup_{r \to t^+} \alpha(r) < 1 \quad \text{for each } t \in [0, \infty),$$

and $\theta \in \Theta_h[0, \Lambda)$. Then,

(1) *For each $p_0 \in \mathcal{X}$, there exists an orbit $\{p_n\}$ of T and $\xi \in \mathcal{X}$ such that $\lim_n p_n = \xi$;*
(2) *ξ is fixed point of T if and only if the function $f(p_1) = d(p_1, Tp_1)$ is T-orbitally lower semi continuous at ξ.*

2. Preliminaries

In 2000 Branciari [4] introduced the notion of v-generalized metric space, for $v = 2$. He called such spaces as generalized metric spaces. The primary goal of Branciari was to generalize Banach-Caccioppoli (Theorem 1.1).

Definition 2.1 ([4, 16, 17])**.** *Let $v \in \mathbb{N}$ and $\mathcal{X}_v$ be a non empty set. A function $d_v : \mathcal{X}_v \times \mathcal{X}_v \to [0, \infty)$ is said to be a v-**generalized metric** on $\mathcal{X}_v$ if for all different $p, u_1, u_2, \ldots, u_v, q$ in $\mathcal{X}_v$ following conditions hold:*

(i) $d_v(p, q) = 0 \Longleftrightarrow p = q$;
(ii) $d_v(p, q) = d_v(q, p)$;
(iii) $d_v(p, q) \leq d_v(p, u_1) + d_v(u_1, u_2) + \cdots + d_v(u_v, q)$.

*The pair $(\mathcal{X}_v, d_v)$ is called v-**generalized metric space**.*

Remark 2.1.

(1) *When $v = 1$, the notion of v-generalized metric space coincides with the notion of a metric space. In this case we write $\mathcal{X}$ instead of $\mathcal{X}_1$ and d in place of of d_1.*
(2) *For $v = 2$, the notion of generalized metric was initiated by Branciari [4]. In this case the inequality (iii) mentioned above is called quadrilateral inequality.*
(3) *For general fixed $v \in \mathbb{N}$, the notion of v-generalized metric appeared in [16, 17], as far as we have seen.*

Branciari[4] introduced the corresponding notions of convergence, Cauchy sequence, completeness and open balls in $\mathcal{X}_2$ in a usual way, as already defined for metric spaces, in literature.

Theorem 2.1 ([4], Theorem 2.1)**.** *Let* $(\mathcal{X}_2, d_2)$ *be a complete* 2*-generalized metric space and* $\lambda \in [0,1)$ *and* $f : \mathcal{X}_2 \to \mathcal{X}_2$ *a mapping such that for each* $p_1, q_1 \in \mathcal{X}_2$ *one has*

$$d_2(f(p_1), f(q_1)) \leq \lambda d_2(p_1, q_1)$$

then

(i) *there exists a point* $a \in \mathcal{X}$ *such that for each* $p_1 \in \mathcal{X}$, $\lim_{n\to\infty} f^n(p_1) = a$*;*
(ii) $f(a) = a$ *and for each* $e \in \mathcal{X}_2$ *such that* $f(e) = e$ *one has* $e = a$*;*
(iii) *for all* $n \in \mathbb{N}$ *one has* $d_2(f^n(p_1), a) \leq \frac{\lambda^n}{1-\lambda} \max\{d_2(p_1, f(p_1)), d_2(p_1, f^2(p_1))\}$.

It was later observed that $(\mathcal{X}_2, d_2)$ metric space may lack basic topological properties: open balls need not be open, d_2 need not be continuous, convergent sequences need not be Cauchy, and limits need not be unique. For clarity, we reproduce the following example from [6].

Example 2.1 ([6], Example 1.1)**.** *Let* $A' = \{0, 2\}$, $B' = \{\frac{1}{m} : m \in \mathbb{N}\}$ *and* $\mathcal{X}_2 = A' \cup B'$. *Define* $d_2 : \mathcal{X}_2 \times \mathcal{X}_2 \to [0, \infty)$ *as follow:*

$$d_2(p_1, q_1) = \begin{cases} 0, & p_1 = q_1 \\ 1, & p_1 \neq q_1 \text{ and } \{p_1, q_1\} \subset A' \text{ or } \{p_1, q_1\} \subset B' \\ q_1, & p_1 \in A',\ q_1 \in B' \\ p_1, & p_1 \in B',\ q_1 \in A'. \end{cases}$$

Then $(\mathcal{X}_2, d_2)$ is a 2-generalized metric space. Note that,

- $\lim_m d_2(\frac{1}{m}, \frac{1}{2}) = 1 \neq d_2(0, \frac{1}{2})$, thus d_2 is not continuous;
- $\lim_m d_2(\frac{1}{m}, 0) = \lim_m d_2(\frac{1}{m}, 2)$, so $\frac{1}{m}$ is not a Cauchy sequence;
- note that for $\epsilon > 0$ we can trace $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$ and if we choose $n \geq N, \frac{1}{n} \in \mathbf{B}(0, \epsilon) \cap \mathbf{B}(2, \epsilon)$, hence, the respective topology is not Hausdorff;
- $\mathbf{B}(\frac{1}{3}, \frac{2}{3}) = \{0, 2, \frac{1}{3}\}$, contains no ball centered at 0, for the reason just mentioned above.

Hence, Theorem 2.1 and several generalizations of Theorem 2.1 obtained soon after Branciari above mentioned result were not completely correct. These issues were noticed independently by Samet [6], Das and Dey [18, 19], and Sarma et al [20], and were discussed in the survey by Kadelburg and Radenovic [21]; see also Kikina and Kikina [22]. To overcome these difficulties, many authors imposed additional conditions, usually Hausdorffness of the induced topology. Jleli and Samet [5] and Turinici [7] showed that such hypotheses can often be replaced by mild adjustments in the statements, making most results valid without extra topological assumptions.

It is a ritualistic approach, for researches working in fixed point theory, to look for multivalued versions of fixed point theorems already proved for singlevalued mappings. The Hausdorff metric (1) plays a pivotal role to obtain such extensions. Unfortunately, if d_v is a generalized metric on $\mathcal{X}_v$ it is not always the case that $\mathcal{H}$ defined by (1) becomes a generalized metric on the set of all nonempty subsets of $\mathcal{X}_v$, as is evident from the following simple example.

Example 2.2 ([21], Example 4.1)**.** *Let $\mathcal{X}_2 = \{p, q, r\}$ and let $d_2 : \mathcal{X}_2 \times \mathcal{X}_2 \to [0, \infty)$ be defined by $d_2(p, q) = 4, d_2(p, r) = d_2(q, r) = 1,$ and $d_2(p_1, q_1) = d_2(q_1, p_1)$ for all $p_1, q_1 \in \mathcal{X}_2$. Then d_2 is a 2-generalized metric space. Note that d_2 is not a metric on $\mathcal{X}_2$. Moreover, for closed and bounded $\{p\}, \{q\}, \{r\}$, and $\{p, r\}$ of $\mathcal{X}_2$ we have*

$$\mathcal{H}\big(\{p\}, \{q\}\big) = 4 > 1+1+1 = \mathcal{H}\big(\{p\}, \{p, r\}\big) + \mathcal{H}\big(\{p, r\}, \{r\}\big) + \mathcal{H}\big(\{r\}, \{q\}\big).$$

In the absence of a metric on the set of subsets of a v-generalized metric ($v \neq 1$), it seems difficult to generalize Nadler's fixed point theorem in the setting of v-generalized metric spaces (for $v \neq 1$) but Suzuki [16], in 2017, elegantly extended Nadler's fixed point theorem for multivalued maps defined on v-generalized metric spaces. Before giving the statement of the theorem established by Suzuki we would like to recollect some essential ingredients from [16, 17].

Definition 2.2 ([16], Definition 3(iii))**.** *Let $\mathcal{X}_v$ be a v-generalized metric space. A sequence $\{p_n\}$ in $\mathcal{X}_v$ is said to be $(\sum, \neq)$-**Cauchy** if p_n are all different and*

$$\sum_{n=1}^{\infty} d_v(p_n, p_{n+1}) < \infty.$$

Definition 2.3 ([17], Definition 4)**.**

(i) *A sequence (p_n) in a v-generalized metric space $(\mathcal{X}_v, d_v)$ is said to be v-**Cauchy** if and only if*

$$\lim_{n\to\infty} \sup\big\{d_v(p_n, p_{n+1+jv}) : j = 0, 1, 2, \cdots\big\} = 0.$$

(ii) *$\mathcal{X}_v$ is said to be v-**complete** if and only if every v-Cauchy sequence converges in $\mathcal{X}_v$.*

Remark 2.2 ([17, 23])**.** *Let $\mathcal{X}_v$ be a v-generalized metric space. Then,*

(i) *A $(\sum, \neq)$-Cauchy sequence is v-Cauchy* [17, Lemma 9].

(ii) *If v is odd then every v-Cauchy sequence in $\mathcal{X}_v$ is 1-Cauchy or simply Cauchy* [17, Proposition 7].

(iii) *If v is even, then every v-Cauchy sequence in $\mathcal{X}_v$ is 2-Cauchy* [17, Proposition 8].

(iv) *When v is even not every $(\sum, \neq)$-Cauchy sequence is Cauchy* [23, Example 1].

Lemma 2.1 ([17], Lemma 12)**.** *Let $\mathcal{X}_v$ be a v-generalized metric space and $\{p_n\}$ be $(\sum, \neq)$-Cauchy sequence in $\mathcal{X}_v$ satisfying either of the following:*

(i) *v is odd and $\mathcal{X}_v$ is complete.*

(ii) *v is even and $\mathcal{X}_v$ is 2-complete.*

Then, there exist $z \in \mathcal{X}_v$ such that $\{p_n\}$ converges only to z.

Now we give the statement of a theorem by Suzuki, in v-generalized metric spaces, which is generalization of the Nadler's fixed point theorem.

Theorem 2.2 ([16], Theorem 20). *Let $(\mathcal{X}_v, d_v)$ be a $(\sum, \neq)$-complete v-generalized metric space and T be a set-valued mapping on $\mathcal{X}_v$ satisfying the following:*

(i) *For any $p_1 \in \mathcal{X}$, Tp_1 is a non-empty subset of $\mathcal{X}_v$.*
(ii) *If a sequence $\{y_n\}$ in Tp_1 converges to q_1 then $q_1 \in Tp_1$ holds.*
(iii) *There exists $\lambda \in [0,1)$ satisfying,*

$$\delta(Tp_1, Tq_1) \leq \lambda d_v(p_1, q_1) \text{ for all } p_1, q_1 \in \mathcal{X}_v, \tag{6}$$

where $\delta(A,B) = \sup_{p\in A} \inf_{q\in B} d_v(p,q)$. Then there exists $z \in \mathcal{X}_v$ satisfying $z \in Tz$.

In 2021, S. Alshaiket et. al. [24] extended Mizoguchi and Takahashi fixed point theorem in v-generalized metric spaces. We state their result for ready reference.

Definition 2.4 ([10]). *A mapping $\alpha : (0,\infty) \to [0,1)$ is called $\mathcal{M}$-$\mathcal{T}$* ***function*** *if*

$$\limsup_{r\to t^+} \alpha(r) < 1 \text{ for each } t \in (0,\infty). \tag{7}$$

Theorem 2.3 ([24], Theorem 3.1). *Let $(\mathcal{X}_v, d_v)$ be a $(\sum, \neq)$-complete v-generalized metric space and T be a set-valued mapping defined from $\mathcal{X}_v$ into $\mathcal{CB}(\mathcal{X}_v)$ satisfies the following:*

(i) *If a sequence $\{y_n\}$ in Tp_1 converges to q_1 then $q_1 \in Tp_1$.*
(ii) *For any $p_1, q_1 \in \mathcal{X}_v$,*

$$\mathcal{H}(Tp_1, Tq_1) \leq \alpha(d_v(p_1, q_1)) d_v(p_1, q_1), \tag{8}$$

where α is $\mathcal{M} - \mathcal{T}$ function, defined by (7). Then, T has a fixed point.

Remark 2.3.

(i) *Note that in above inequality $\mathcal{H}$ is not a metric on $\mathcal{CB}(\mathcal{X}_v)$.*
(ii) *In view of hypothesis (i) of Theorem 2.3, T may assume its values in $\mathcal{N}(\mathcal{X}_v)$ instead of $\mathcal{CB}(\mathcal{X}_v)$.*

The purpose of this paper is to prove a Mizoguchi and Takahashi type fixed point theorem in v-generalized metric spaces. Our results generalize Theorem 1.7, Theorem 2.2, Theorem 2.3 and those contains therein. Non trivial examples are provided to support our claims.

3. **Main results**

We begin this section with the following lemma. The proof is straight forward, similar to [15, Lemma 2.2].

Lemma 3.1. *Let $\mathcal{B}$ be a nonempty subset of a v-generalized metric space $(\mathcal{X}_v, d_v)$ and $\mu > 1$. If for $p_1 \in \mathcal{X}_v, d_v(p_1, \mathcal{B}) > 0$ then there exists $b \in \mathcal{B}$ such that*

$$d_v(p_1, b) \leq \mu d_v(p_1, \mathcal{B}).$$

Now we introduce the following definition, it is weaker than the corresponding Definition 2.2.

Definition 3.1. *Let $\mathcal{X}_v$ be a v-generalized metric space and T be a mapping from $\mathcal{X}_v$ into $\mathcal{N}(\mathcal{X}_v)$. A sequence $\{p_n\}$ in $\mathcal{X}_v$ is said to be $(\sum, \neq)$ T-**orbitally Cauchy** at p_0 if there exist an orbit $O(p_0, T)$ of T at p_0 such that all $p_n \in O(p_0, T)$ (Definition 1.4) are different and*

$$\sum_{n=1}^{\infty} d_v(p_n, p_{n+1}) < \infty.$$

Example 3.1. *Let $\mathcal{X} = [0, 3]$, Define $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ as follows:*

$$d(p, q) = \begin{cases} 0 \text{ if } p = q, \\ 1 \text{ if } p \neq q. \end{cases}$$

Then $(\mathcal{X}, d)$ is a 1-generalized metric space or simply a metric space. Define $T : \mathcal{X} \to \mathcal{CL}(\mathcal{X})$ by

$$T(p) = \begin{cases} \{\frac{p}{2}\} \text{ if } p \in [0, 2), \\ \{\frac{p+1}{2}\} \text{ if } p \in [2, 3]. \end{cases}$$

Assume that $p_0 = 1 \in \mathcal{X}$. Then, we have

$$O(1, T) = \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \right\} = \left\{\frac{1}{2^n}; n = 0, 1, 2, \ldots, \right\} = p_n.$$

$$d(p_n, p_{n+1}) = \left|\frac{1}{2^n} - \frac{1}{2^{n+1}}\right| = \frac{1}{2^{n+1}}.$$

Note that p_n are all distinct. The series $\sum_{n=1}^{\infty} \frac{1}{2^{n+1}}$ is convergent and by comparison test $\sum_{n=1}^{\infty} d(p_n, p_{n+1})$ is convergent. Thus p_n is $(\sum, \neq)$ T-orbitally Cauchy sequence at p_0.

Definition 3.2. *A mapping $T : \mathcal{X}_v \to \mathcal{N}(\mathcal{X}_v)$ is called **Mizoguchi-Takahashi's θ-type v-contraction** if*

$$\theta(d_v(q_1, Tq_1)) \leq \alpha(d_v(p_1, q_1))\theta(d_v(p_1, q_1)) \text{ for } p_1 \in \mathcal{X}_v \text{ and } q_1 \in Tp_1, \quad (9)$$

where α is a $\mathcal{M}$-$\mathcal{T}$ function (Definition 2.4) and $\theta \in \Theta_h[0, \Lambda)$.

Remark 3.1. *Note that the inequality (9) is weaker than the corresponding inequality (6) in Theorem 2.2. We give following example in support of our claim.*

Example 3.2. *Let* $A_1 = \{2^{-n} : n \in \mathbb{N}\}$, $B_1 = \{0, 1\}$ *and* $\mathcal{X}_2 = A_1 \cup B_1$. *Define* $d_2 : \mathcal{X}_2 \times \mathcal{X}_2 \to \mathbb{R}$ *as follows:*

$$d_2(q_1, p_1) = d_2(p_1, q_1) = \begin{cases} 0 \text{ if } p_1 = q_1, \\ p_1 \text{ if } p_1 \in A_1 \text{ and } q_1 \in B_1, \\ 1 \text{ otherwise.} \end{cases}$$

Then $(\mathcal{X}_2, d_2)$ *is a 2-generalized metric space but not a standard metric space because*

$$1 = d_2(\frac{1}{2}, \frac{1}{4}) > d_2(\frac{1}{2}, 1) + d_2(1, \frac{1}{4}) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

Define $T : \mathcal{X}_2 \to \mathcal{CL}(\mathcal{X}_2)$ *by*

$$T(p_1) = \begin{cases} \{\frac{1}{8}, 1\} \text{ if } p_1 = \frac{1}{4}, \\ \{\frac{1}{4}, \frac{1}{16}\} \text{ if } p_1 = \frac{1}{2}, \\ \{1\} \text{ if } p_1 \in \mathcal{X}_2 \setminus \{\frac{1}{2}, \frac{1}{4}\}. \end{cases}$$

Moreover, define $\alpha : (0, \infty) \to [0, 1)$ *and* $\theta \in \Theta_h[0, \Lambda)$ *by*

$$\alpha(t) = \begin{cases} \frac{t}{1+t} \text{ for } t \in [0, 1), \\ \frac{1}{2} \text{ otherwise.} \end{cases} \qquad \theta(t) = \begin{cases} t^{\frac{1}{2}} \text{ for } t \in [0, 1], \\ \frac{t}{2} \text{ otherwise.} \end{cases}$$

We see that

$$\theta(d_2(q_1, Tq_1)) \leq \alpha(d_2(p_1, q_1))\theta(d_2(p_1, q_1)),$$

for $p_1 \in \mathcal{X}_2$ *and* $q_1 \in Tp_1$. *Note that inequality (6) does not hold if we take* $p_1 = \frac{1}{4}$ *and* $q_1 = \frac{1}{2}$.

Next lemma is extracted from the proof of [15, Theorem 4.2]

Lemma 3.2. *Let* $\mathcal{X}_v$ *be a* v*-generalized metric space and* $T : \mathcal{X}_v \to \mathcal{CL}(\mathcal{X}_v)$ *is a Mizoguchi-Takahashi's* θ*-type* v*-contraction. Assume that* T *has no fixed point. Then, there exists a* $(\sum, \neq)$ T*-orbitally Cauchy sequence at* $p \in \mathcal{X}_v$.

Proof. Suppose that $p_0 \in \mathcal{X}_v$. Since $Tp_0 \neq \emptyset$, there exists $p_1 \in \mathcal{X}_v$ such that $p_1 \in Tp_0$. Assume that T has no fixed point then $p_0 \neq p_1$ thus $d(p_0, p_1) > 0$. Let $\mu_{0,1} = \frac{1}{\sqrt{\alpha(d_v(p_0,p_1))}}$. Observe that $\mu_{0,1} > 1$. Note that $d_v(p_1, Tp_1) > 0$, otherwise $p_1 \in \overline{Tp_1} = Tp_1$ and p_1 becomes fixed point of T. Since θ is nondecreasing and positive homogeneous (Definitions 1.2(i), 1.3), by using (5) it follows from Lemma 3.1 that there exists an element $p_2 \in Tp_1$ such that $p_1 \neq p_2$ and

$$\theta(d_v(p_1, p_2)) \leq \mu_{0,1}\ \theta(d_v(p_1, Tp_1)).$$

Repeating the above argument, since $\mu_{1,2} = \frac{1}{\sqrt{\alpha(d_v(p_1,p_2))}} > 1$ and $Tp_2 \neq \emptyset$, it follows that there is an element $p_3 \in Tp_2$ such that $p_2 \neq p_3$ and

$$\theta(d_v(p_2, p_3)) \leq \mu_{1,2}\ \theta(d_v(p_2, Tp_2)).$$

In this way we obtain a sequence $\{p_n\} \subset \mathcal{X}_v$ such that $p_n \in Tp_{n-1}$ and p_n are all different. Moreover,

$$\theta(d_v(p_n, p_{n+1})) \leq \mu_{n-1,n}\, \theta(d_v(p_n, Tp_n)), \tag{10}$$

where $\mu_{n-1,n} = \frac{1}{\sqrt{\alpha(d_v(p_{n-1},p_n))}} > 1$.
Using (9) it follows from (10) that

$$\begin{aligned} \theta(d_v(p_n, p_{n+1})) &\leq \frac{1}{\mu_{n-1,n}}\theta(d_v(p_{n-1}, p_n)) \\ &< \theta(d_v(p_{n-1}, p_n)). \end{aligned} \tag{11}$$

Hence $\{\theta(d_v(p_n, p_{n+1}))\}$ is a monotonically decreasing sequence. Therefore, it converges to its infimum, $u \geq 0$. We claim that $u = 0$. Otherwise, for $u > 0$, from (11), we have

$$u \leq \sqrt{\lim_{n\to\infty} \sup \alpha(d_v(p_{n-1}, p_n))}u < u,$$

a contradiction.

Since θ is strictly inverse isotone (see inequality (3)), $\{d_v(p_n, p_{n+1})\}$ is also a decreasing sequence of positive real numbers bounded below by 0, and thus convergent. We claim that $\{d_v(p_n, p_{n+1})\}$ also converges to 0. Suppose $d_v(p_n, p_{n+1}) \to c > 0$. Then, for $0 < \epsilon < c$, there exist a natural number m such that

$$0 < \delta = c - \epsilon < (d_v(p_n, p_{n+1}))\ \forall\ n \geq m.$$

Since θ is positive and nondecreasing,

$$0 < \theta(\delta) \leq \{\theta(d_v(p_n, p_{n+1}))\}\ \forall\ n \geq m,$$

which is contradiction, since $\{\theta(d_v(p_n, p_{n+1}))\} \to 0$. From (11) , we get

$$\theta(d_v(p_n, p_{n+1})) \leq \left[\sqrt{\alpha(d_v(p_{n-1}, p_n))} \ldots \sqrt{\alpha(d_v(p_0, p_1))}\ \right]\theta(d_v(p_0, p_1)). \tag{12}$$

Let $\gamma > 0$ and $0 < \delta < 1$, it follows from (7) that for $t \in (0, \gamma)$

$$\alpha(t) < \delta^2.$$

Since $d_v(p_{n-1}, p_n) \to 0$, for above $\gamma > 0$, one may find $N \in \mathbb{N}$ such that

$$d_v(p_{n-1}, p_n) < \gamma \text{ for } n \geq N.$$

Then, from (12) we have

$$\begin{aligned} \theta(d_v(p_n, p_{n+1})) &\leq \delta^{n-(N-1)}[\sqrt{\alpha(d_v(p_{N-2}, p_{N-1}))} \ldots \sqrt{\alpha(d_v(p_0, p_1))}]\theta(d_v(p_0, p_1)) \\ &< \delta^n \frac{\theta(d_v(p_0, p_1))}{\delta^{N-1}}. \end{aligned}$$

Note that $\delta < 1$. Therefore the series $\sum_{n=1}^{\infty} \delta^n \frac{\theta(d_v(p_0,p_1))}{\delta^{N-1}}$ is convergent and by comparison test, $\sum_{n=1}^{\infty} \theta(d_v(p_n, p_{n+1}))$ is convergent, then $\sum_{n=1}^{\infty}(d_v(p_n, p_{n+1}))$ is convergent. Thus $\{p_n\}$ is $(\sum, \neq)$ T-orbitally Cauchy sequence at p_0. $\square$

Theorem 3.1. *Let $\mathcal{X}_v$ be a v-generalized metric space and $T : \mathcal{X}_v \to \mathcal{CL}(\mathcal{X}_v)$ is a Mizoguchi-Takahashi's θ-type v-contraction. Assume that the function $f(p) = d_v(p, Tp)$ is T-orbitally lower semi continuous and either of the following holds:*

(i) *v is odd and $\mathcal{X}_v$ is complete;*
(ii) *v is even and $\mathcal{X}_v$ is 2-complete.*

Then, T has a fixed point.

Remark 3.2.

- *Our main result assumes that the mapping T has values in closed subsets of the v-generalized metric space $\mathcal{X}_v$ but in Theorem 2.2 the range of the mapping is just nonempty subset of $\mathcal{X}_v$. Note that if we look closely at condition (ii) of Theorem 2.2 then we observe that Tp is closed in $\mathcal{X}_v$.*
- *If T takes values in $\mathcal{N}(\mathcal{X}_v)$ instead of $\mathcal{CL}(\mathcal{X}_v)$ and condition (ii) of Theorem 2.2 holds then it follows from above Theorem as well that T has a fixed point.*
- *Moreover, as mentioned already, the inequality (9) is weaker than the corresponding inequality (6) in Theorem 2.2.*
- *Note that if $q_1 \in Tp_1$ then $d_v(q_1, Tq_1) \leq \mathcal{H}(Tp_1, Tq_1)$. Taking $\theta(t) = t$ we see that the inequality (8) is a special case of the inequality (9). Therefore, Theorem 2.3 is a special case of Theorem 3.1.*
- *Thus our result is indeed true generalization of not only Theorem 1.7 but also for Theorem 2.2 and Theorem 2.3 and those contained therein.*

We now give proof of Theorem 3.1.

Proof. Let $p_0 \in \mathcal{X}_v$. Suppose that T has no fixed point then it follows from Lemma 3.2 that there exists a $(\sum, \neq)$ T-orbitally Cauchy sequence at p_0. Further, assume that either of the hypothesis (i) or (ii) holds then it follows from Lemma 2.1, that there is $\xi \in \mathcal{X}_v$ such that $p_n \to \xi$. Since $p_n \in Tp_{n-1}$, letting $n \to \infty$ from (9) and using condition (θ is strictly inverse isotone, inequality (3)) we have $d_v(p_n, Tp_n) \to 0$. Moreover, the T-orbitally lower continuity of the function $p \mapsto d_v(p, Tp)$ at ξ implies that $d_v(\xi, T\xi) = 0$. Furthermore, $\xi \in T\xi$, since $T\xi$ is closed, a contradiction. Thus, T has a fixed point. $\square$

The following example substantiates our claim.

Example 3.3. *Let $\mathcal{X}_2 = \{n; n \in \mathbb{N}\} \cup \{0\}$. Define $d_2 : \mathcal{X}_2 \times \mathcal{X}_2 \to \mathbb{R}$ as follows:*

$$d_2(q_1, p_1) = d_2(p_1, q_1) = \begin{cases} 0 \text{ if } p_1 = q_1, \\ \frac{1}{2} \text{ if } p_1 \in \{n : n \in \mathbb{N}\} \text{ and } q_1 = 0,\ p_1 \neq\ q_1 \\ \frac{1}{p_1} \text{ if } p_1 \in \{2n : n \in \mathbb{N}\} \text{ and } q_1 \in \{2n - 1 \in \mathbb{N}\}, \\ 1 \text{ otherwise.} \end{cases}$$

Then $(\mathcal{X}_2, d_2)$ *is a 2-generalized metric space but not a standard metric space because*

$$1 = d_2(2,4) > d_2(2,3) + d_2(3,4) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

Define $T : \mathcal{X}_2 \to \mathcal{CL}(\mathcal{X}_2)$ *by*

$$T(p_1) = \begin{cases} \{1, 20\} \text{ if } p_1 \in \{0, 2, 3\}, \\ \{3, 20\} \text{ if } p_1 \in \{2n-1 : n \in \mathbb{N}\} \setminus \{3\}, \\ \{2, 20\} \text{ if } p_1 \in \{2n : n \in \mathbb{N}\} \setminus \{2\}. \end{cases}$$

Moreover, let $\alpha : (0, \infty) \to [0, 1)$ *and* $\theta : [0, \Lambda) \to \mathbb{R}$ *are defined by*

$$\alpha(t) = \begin{cases} \frac{1}{2} \text{ for } t = \frac{1}{20}, \\ \frac{\sqrt{10}}{10} \text{ for } t = \frac{1}{2}, \\ \frac{\sqrt{2}}{2} \text{ for } t = 1, \\ \frac{2}{3} \text{ otherwise.} \end{cases} \qquad \theta(t) = \begin{cases} t^{\frac{1}{2}} \text{ for } t \in [0, 1], \\ \frac{t}{2} \text{ otherwise.} \end{cases}$$

For $p_1 \in \mathcal{X}_2$ *and* $q_1 \in Tp_1$ *we define*

$$\eta(q_1) = d_2(q_1, Tq_1) \text{ and } \omega(p_1, q_1) = \alpha(d_2(p_1, q_1))\theta(d_2(p_1, q_1)).$$

We claim that

$$\eta(q_1) \le \omega(p_1, q_1).$$

There are ten cases to be discussed, we discuss them one by one.

(i) *For* $p_1 = 0,\ q_1 = 1$ *we have* $\frac{\sqrt{5}}{10} = \eta(q_1) \le \omega(p_1, q_1) = \frac{\sqrt{5}}{10}$.

(ii) *For* $p_1 = 0,\ q_1 = 20$ *we have* $0 = \eta(q_1) \le \omega(p_1, q_1) = \frac{\sqrt{5}}{10}$.

(iii) *For* $p_1 = 2,\ q_1 = 1$ *we have* $\frac{\sqrt{5}}{10} = \eta(q_1) \le \omega(p_1, q_1) = \frac{\sqrt{5}}{10}$.

(iv) *For* $p_1 = 2,\ q_1 = 20$ *we have* $0 = \eta(q_1) \le \omega(p_1, q_1) = \frac{\sqrt{2}}{2}$.

(v) *For* $p_1 = 3,\ q_1 = 1$ *we have* $\frac{\sqrt{5}}{10} = \eta(q_1) \le \omega(p_1, q_1) = \frac{\sqrt{2}}{2}$.

(vi) *For* $p_1 = 3,\ q_1 = 20$ *we have* $0 = \eta(q_1) \le \omega(p_1, q_1) = \frac{\sqrt{5}}{20}$.

(vii) *For* $p_1 \in \{2n-1 : n \in \mathbb{N}\} \setminus \{3\},\ q_1 = 3$ *we have*

$$\frac{\sqrt{5}}{10} = \eta(q_1) \le \omega(p_1, q_1) = \frac{\sqrt{2}}{2}.$$

(viii) *For* $p_1 \in \{2n-1 : n \in \mathbb{N}\} \setminus \{3\},\ q_1 = 20$ *we have*

$$0 = \eta(q_1) \le \omega(p_1, q_1) = \frac{\sqrt{5}}{20}.$$

(ix) *For* $p_1 \in \{2n : n \in \mathbb{N}\} \setminus \{2\},\ q_1 = 2$ *we have*

$$\frac{\sqrt{2}}{2} = \eta(q_1) \le \omega(p_1, q_1) = \frac{\sqrt{2}}{2}.$$

(x) *For* $p_1 \in \{2n : n \in \mathbb{N}\} \setminus \{2\},\ q_1 = 20$ *we have* $0 = \eta(q_1) \le \omega(p_1, q_1)$.

Looking at the points of the space X_2 and the definition of the metric d_2, we observe that distinct terms of a sequence p_n can't be arbitrary close. Therefore, a Cauchy sequence in X_2 is always constant. Moreover, any arbitrary convergent sequence in an orbit of T is always constant. Therefore, X_2 is 2-complete and the function f is T orbitally lower semicontinuous. Thus, all hypothesis of Theorem 3.1 are satisfied and 20 is fixed point of T. Note that Theorem 2.2 is not applicable here since inequality (6) does not hold if we take $p_1 = 1$ and $q_1 = 20$.

4. Conclusion and Future Directions

To say that Theorem A is a special Theorem B, we write Theorem A $\rightarrow$ Theorem B. Using this terminology we have the following inclusions:

$$\begin{array}{lcl}
1930 : \text{Theorem } 1.1 & \rightarrow & 2000 : \text{Theorem } 2.1 \\
\qquad\quad \downarrow & & \qquad\quad \downarrow \\
1969 : \text{Theorem } 1.2 & \rightarrow & 2017 : \text{Theorem } 2.2 \\
\qquad\quad \downarrow & & \\
1972 : \text{Theorem } 1.3 & & \qquad\quad \downarrow \\
\qquad\quad \downarrow & & \\
1989 : \text{Theorem } 1.4 & \rightarrow & 2021 : \text{Theorem } 2.3 \\
\qquad\quad \downarrow & & \\
2007 : \text{Theorem } 1.5 & & \\
\qquad\quad \downarrow & & \qquad\quad \downarrow \\
2009 : \text{Theorem } 1.6 & & \\
\qquad\quad \downarrow & & \\
2011 : \text{Theorem } 1.7 & \rightarrow & \text{Main Result} : \text{Theorem } 3.1
\end{array}$$

Bakhtin [25] introduced the notion of quasimetric in 1989, which was later called b-metric by Czerwik [26] in 1998. Kamran [27, 28] et. al. generalized the notion of b-metric by introducing the notion of extended b-metric in 2017. In 2018 Mlaiki [29, 30] and his coauthors further extended the notion of extended b-metric by introducing the idea of controlled and double controlled metric spaces. Some significant results have also been discussed in [31, 32]. We belief that all these results can be extended for multivalued maps in v-generalized metric spaces using our technique.

Acknowledgment

We are grateful to the reviewers for their valuable comments. They help us to improve the manuscript substantially. Further, the authors are thankful to the Deanship of Scientific Research, Islamic University of Madinah for providing the support under the Post Publication Program V.

REFERENCES

[1] *S. Banach*, Sur les operations dans les ensembles abstraits et leur application aux equations integrales, Fund. Math., **3**(1922), No. 1, 133–181.

[2] *R. Caccioppoli*, Un teorema generale sull esistenza di elementi uniti in una transformazione funzionale, Rend. Accad. Naz. Lincei, **11**(1930), 794–799.

[3] *S.B. Nadler Jr.*, Multivalued contraction mappings, Pacific J. Math., **30**(1969), 475–488.

[4] *A. Branciari*, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, Publ. Math. Debrecen, **57**(2000), No. 1–2, 31–37.

[5] *M. Jleli and B. Samet*, The Kannan's fixed point theorem in a cone rectangular metric space, J. Nonlinear Sci. Appl., **2**(2009), No. 3, 161–167.

[6] *B. Samet*, Discussion on: a fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces by A. Branciari, Publ. Math. Debrecen, **76**(2010), No. 4, 493–494.

[7] *M. Turinici*, Functional contractions in local Branciari metric spaces, arXiv preprint arXiv:1208.4610, 2012.

[8] *R.M. Brooks and K. Schmitt*, The contraction mapping principle and some applications, Electron. J. Differ. Equ., **2009**(2009), 1–90.

[9] *J.R. Graef, J. Henderson and A. Ouahab*, Impulsive differential inclusions: a fixed point approach, Vol. 20, Walter de Gruyter, 2013.

[10] *S. Reich*, Fixed points of contractive functions, Boll. Unione Mat. Ital., **4**(1972), No. 5, 26–42.

[11] *N. Mizoguchi and W. Takahashi*, Fixed point theorems for multivalued mappings on complete metric spaces, J. Math. Anal. Appl., **141**(1989), No. 1, 177–188.

[12] *D. Klim and D. Wardowski*, Fixed point theorems for set-valued contractions in complete metric spaces, J. Math. Anal. Appl., **334**(2007), No. 1, 132–139.

[13] *Y. Feng and S. Liu*, Fixed point theorems for multi-valued operators, Fixed Point Theory Appl., **2007**(2007), Article ID 34248.

[14] *H.K. Pathak and N. Shahzad*, Fixed point results for set-valued contractions by altering distances in complete metric spaces, Nonlinear Anal., **70**(2009), No. 7, 2634–2641.

[15] *T. Kamran and Q. Kiran*, Fixed point theorems for multi-valued mappings obtained by altering distances, Math. Comput. Modelling, **54**(2011), No. 11–12, 2772–2777.

[16] *T. Suzuki*, Nadler fixed point theorem in v-generalized metric spaces, Fixed Point Theory Appl., **2017**(2017), Article ID 18.

[17] *B. Alamri, T. Suzuki and L.A. Khan*, Caristi fixed point theorem and Subrahmanyam fixed point theorem in v-generalized metric spaces, J. Funct. Spaces, **2015**(2015), Article ID 709391.

[18] *P. Das and L.K. Dey*, Fixed point of contractive mappings in generalized metric spaces, Math. Slovaca, **59**(2009), 499–504.

[19] *P. Das and L.K. Dey*, Porosity of certain classes of operators in generalized metric spaces, Demonstratio Math., **42**(2009), No. 1, 163–174.

[20] *I.R. Sarma, J.M. Rao and S.S. Rao*, Contractions over generalized metric spaces, J. Nonlinear Sci. Appl., **2**(2009), No. 3, 180–182.

[21] *Z. Kadelburg and S. Radenovic*, On generalized metric spaces: a survey, TWMS J. Pure Appl. Math., **5**(2014), No. 1, 3–13.

[22] *L. Kikina and K. Kikina*, Fixed point theorems on generalized metric spaces for mappings in a class of almost f-contractions, Demonstratio Math., **48**(2015), No. 3, 440–451.

[23] *W.A. Kirk and N. Shahzad*, Correction: Generalized metrics and Caristi's theorem, Fixed Point Theory Appl., **2014**(2014), Article ID 177.

[24] *S. Alshaikey, S.M. Alsulami and M. Alansari*, Mizoguchi-Takahashi's fixed point theorem in v-generalized metric spaces, Journal of Computational Analysis and Applications, **2021**(2021), Article ID 1047.

[25] *I. Bakhtin*, The contraction mapping principle in quasimetric spaces, Functional analysis, **30**(1989), 26–37.

[26] *S. Czerwik*, Nonlinear set-valued contraction mappings in b-metric spaces, Atti Sem. Mat. Fis. Univ. Modena, **46**(1998), 263–276.

[27] *T. Kamran, M. Samreen, and Q.UL Ain*, A generalization of b-metric space and some fixed point theorems. Mathematics, **5**(2017), No. 2, Article ID 19.

[28] *M. Samreen, T. Kamran and M. Postolache*, Extended b-metric space, extended b-comparison function and nonlinear contractions, Sci. Bull. Ser. A Appl. Math. Phys., **80**(2018), No. 4, 21–28.

[29] *N. Mlaiki, H. Aydi, N. Souayah, and T. Abdeljawad*, Controlled metric type spaces and the related contraction principle, Mathematics, **6**(2018), No. 10, Arrticle ID 194.

[30] *T. Abdeljawad, N. Mlaiki, H. Aydi, N. Souayah*, Double controlled metric type spaces and some fixed point results, Mathematics, **6**(2018) No. 12, Article ID 320.

[31] *A. E. Al-Mazrooei, and J. Ahmad*, Fixed Point Results in Controlled Metric Spaces with Applications, Mathematics, **10**(2022), No. 3, Article ID 490.

[32] *H. Aydi, H. Hammouda, and S. Mansour*, A fixed point result in a double control metric space setting and a Fredholm integral equation, Eur. J. Pure Appl. Math., **18**(2025), No. 3, Article ID 6133.