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ON A NATURAL YANG-MILLS SURFACE PRODUCED BY THE RABINOVICH-FABRIKANT DYNAMICAL SYSTEM

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Using the least squares variational method, we construct the Lagran-ge-Hamilton geometry (in the sense of Lagrangian nonlinear connection, Lagrangian d-torsions, Lagrangian Yang-Mills electromagnetic-like energy, Hamiltonian nonlinear connection and Hamiltonion d-torsions) determined by the important Rabinovich-Fabrikant dynamical system which characterizes the perturbations in the structure of bridges and airplane wings. The constant level Yang-Mills electromagnetic-like surfaces are investigated. Also natural parametrizations of these surfaces, together with their first fundamental forms and geodesic equations, are presented.

Keywords: Rabinovich-Fabrikant dynamical system, (co)tangent bundles, least squares Lagrangian and Hamiltonian, Lagrange-Hamilton geometry
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1. **Introduction**

In the study of dynamical systems (Böhmer et al. [2]), a very important role is played by the Jacobi stability (which establishes whether their trajectories converge, diverge or are chaotic). After the development of the Lagrange-Hamilton geometry (Miron at el. [6], [7], [3]), together with the variational least squares method initiated by Udriște and Neagu in [12], [11], [8], an important geometrical instrument was created for the study of the Jacobi stability of dynamical systems (e.g., Neagu-Ovsiyuk [9]). These geometrical ideas were applied to the Rabinovich-Fabrikant dynamical system by Neagu and Ovsiyuk in [10], which describe the surfaces of constant level attached to the Lagrangian Yang-Mills energy associated with this dynamical system. For such a reason, in this paper we describe a natural parametrization for these

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surfaces (illustrated by plots), which allows us to compute the first fundamental form and the equation of their geodesics. In our opinion, these bring a novelty for the investigation of the initial Rabinovich-Fabrikant dynamical system, which could provide practical relevance.

In such a context, we recall that the Rabinovich-Fabrikant dynamical system is defined by ($\gamma,\ \nu > 0$ are some given parameters) [4]

$$\begin{cases} \dfrac{dx}{dt} = y(z-1+x^2)+\gamma x \\ \dfrac{dy}{dt} = x(3z+1-x^2)+\gamma y \\ \dfrac{dz}{dt} = -2z(\nu+xy). \end{cases} \tag{1}$$

1.1. **The Lagrange geometry and Jacobi stability problem produced by the Rabinovich-Fabrikant dynamical system**

In the context of the Rabinovich-Fabrikant dynamical system (1) we work on the particular 3-dimensional manifold $M = \mathbb{R}^3$, whose coordinates are $(x^1, x^2, x^3) = (x, y, z)$, and with corresponding coordinates for their tangent TM and cotangent T^*M bundles given by $(x^i, y^i)_{i=\overline{1,3}}$ and $(x^i, p_i)_{i=\overline{1,3}}$, respectively.

Firstly, consider the vector field $X = (X^i(x, y, z))_{i=\overline{1,3}}$ on the manifold $M = \mathbb{R}^3$, given by

$$X^1(x,y,z) = y(z-1+x^2)+\gamma x, \quad X^2(x,y,z) = x(3z+1-x^2)+\gamma y,$$
$$X^3(x,y,z) = -2z(\nu+xy),$$

which allows us to regard the Rabinovich-Fabrikant dynamical system (1) as the system

$$\frac{dx^i}{dt} = X^i(x(t)),\ i = \overline{1,3}. \tag{2}$$

Note that the system (1) is used in Engineering and Physics, being related to unexpected and potential responses to perturbations in a structure such as a bridge or airplane wing (Gupta and Yadav [4]).

The solutions of class C^2 of the dynamical system (2) are the global minimum points for the *least squares Lagrangian* $L : TM \to \mathbb{R}$ which is defined by[1]

$$\begin{gathered} L(x,y) = \delta_{ij}\left(y^i - X^i(x)\right)\left(y^j - X^j(x)\right) \Leftrightarrow \\ L(x,y) = \left(y^1 - X^1(x)\right)^2 + \left(y^2 - X^2(x)\right)^2 + \left(y^3 - X^3(x)\right)^2 \geq 0. \end{gathered} \tag{3}$$

The Euler-Lagrange equations of the least squares Lagrangian (3) are

$$\frac{\partial L}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial y^k}\right) = 0, \quad y^k = \frac{dx^k}{dt}, \quad k = \overline{1,3},$$

and can be rewritten in the following geometrical form:

$$\frac{d^2x^k}{dt^2} + 2G^k(x,y) = 0, \quad k = \overline{1,3}, \tag{4}$$

where the object $G = (G^k)_{k=\overline{1,3}}$ described by

$$G^k = \frac{1}{2}\left(\frac{\partial^2 L}{\partial x^j \partial y^k} y^j - \frac{\partial L}{\partial x^k}\right) = -\frac{1}{2}\left[\left(\frac{\partial X^k}{\partial x^j} - \frac{\partial X^j}{\partial x^k}\right) y^j + \frac{\partial X^j}{\partial x^k} X^j\right]$$

is equipped with the geometrical meaning of *semispray* of L.

Remark 1.1. *We remind that, under a transformation of coordinates* $\tilde{x}^i = \tilde{x}^i(x^j)$, *the* ***semispray*** $G^i(x,y)$ *changes as follows* [6], [1]*:*

$$2\tilde{G}^k = 2G^j \frac{\partial \tilde{x}^k}{\partial x^j} - \frac{\partial x^i}{\partial \tilde{x}^j}\frac{\partial \tilde{y}^k}{\partial x^i}\tilde{y}^j.$$

It is already a well-known fact that, according to the geometrical concepts from the works Miron and Anastasiei [6], Udrişte and Neagu [12], [11], and Balan and Neagu [1], we can construct a complete natural collection of nonzero Lagrangian geometrical objects (such as nonlinear connection, distinguished (d-) torsions and Yang-Mills electromagnetic-like energy) that characterize the dynamical system (2) and further, the Rabinovich-Fabrikant system (1).

We note that the Jacobian matrix $J = \left(\frac{\partial X^i}{\partial x^j}\right)_{i,j=\overline{1,3}}$ of the vector field $X(x,y,z)$ is expressed by the matrix

$$J = \begin{pmatrix} 2xy + \gamma & z - 1 + x^2 & y \\ 3z + 1 - 3x^2 & \gamma & 3x \\ -2yz & -2xz & -2(\nu + xy) \end{pmatrix}.$$

Then tracing the ideas from [10], we obtain the following result:

[1]The Latin indices $i, j, k, ...$ run from 1 to 3. Additionally, the Einstein convention of summation is adopted all over this work.

Proposition 1.1. *The* ***Lagrangian canonical nonlinear connection*** *on the tangent bundle TM, produced by the Rabinovich-Fabrikant dynamical system (1), is described by the skew-symmetric matrix*

$$N = \begin{pmatrix} 0 & z+1-2x^2 & -y/2-yz \\ -z-1+2x^2 & 0 & -3x/2-xz \\ y/2+yz & 3x/2+xz & 0 \end{pmatrix}.$$

Proof. The entries of the canonical nonlinear connection $N = \left(N^i_j\right)_{i,j=\overline{1,3}}$ are defined by the formulas $N^i_j = \partial G^i/\partial y^j$ (see [6]). By direct computation, this implies the following matrix relation (see [1]): $N = -(1/2)\left[J - J^t\right]$. □

Remark 1.2. *By changing the coordinates as $\tilde{x}^i = \tilde{x}^i(x^j)$, the* ***Lagrangian nonlinear connection*** *components $N^i_j(x,y)$ transform on TM as* [6], [1]*:*

$$\tilde{N}^k_l = N^j_i \frac{\partial x^i}{\partial \tilde{x}^l}\frac{\partial \tilde{x}^k}{\partial x^j} - \frac{\partial x^i}{\partial \tilde{x}^l}\frac{\partial \tilde{y}^k}{\partial x^i}.$$

Moreover, the Cartan canonical linear connection on the tangent bundle TM induced by the least squares Lagrangian (3) has all its adapted components equal to zero.

Definition 1.1. *It is well known that, with respect to the transformation of coordinates $\tilde{x}^i = \tilde{x}^i(x^j)$, a* ***Lagrangian d-tensor*** *$T^{ij\ldots}_{kl\ldots}(x,y)$ on the tangent bundle TM obey the rules of transformation* [6], [1]*:*

$$T^{ij\ldots}_{kl\ldots} = \tilde{T}^{pm\ldots}_{rs\ldots}\frac{\partial x^i}{\partial \tilde{x}^p}\frac{\partial x^j}{\partial \tilde{x}^m}\frac{\partial \tilde{x}^r}{\partial x^k}\frac{\partial \tilde{x}^s}{\partial x^l}\cdots.$$

Remark 1.3. *Notice that the Cartan canonical linear connection is characterized by some distinguished nonzero torsions.*

As a result, we use the assertion from [10]:

Proposition 1.2. *The Lagrangian canonical Cartan linear connection, produced by the Rabinovich-Fabrikant dynamical system (1), is characterized by the* ***d-torsion skew-symmetric matrices****:*

$$R_1 = \frac{\partial N}{\partial x^1} = \frac{\partial N}{\partial x}, \quad R_2 = \frac{\partial N}{\partial x^2} = \frac{\partial N}{\partial y}, \quad R_3 = \frac{\partial N}{\partial x^3} = \frac{\partial N}{\partial z},$$

where

$$R_1 = \begin{pmatrix} 0 & -4x & 0 \\ 4x & 0 & -t_1 \\ 0 & t_1 & 0 \end{pmatrix}, \; R_2 = \begin{pmatrix} 0 & 0 & -t_2 \\ 0 & 0 & 0 \\ t_2 & 0 & 0 \end{pmatrix}, \; R_3 = \begin{pmatrix} 0 & 1 & -y \\ -1 & 0 & -x \\ y & x & 0 \end{pmatrix},$$

where $t_1 = 3/2 + z$ *and* $t_2 = 1/2 + z$.

Proof. The general formulas which define the Lagrangian d-torsions are expressed by (see [6])

$$R_k = \left(R^i_{jk} := \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j} \right)_{i,j=\overline{1,3}},$$

where $\delta/\delta x^k = \partial/\partial x^k - N^r_k \partial/\partial y^r$. Through direct computations, we get the following matrix relations (see [1]): $R_k = \partial N/\partial x^k$, $\forall\ k = \overline{1,3}$. □

Proposition 1.3. *The* ***Lagrangian Yang-Mills electromagnetic-like energy****, produced by the Rabinovich-Fabrikant dynamical system (1), is given by*

$$\mathcal{EYM}(x) = \frac{1}{2} \cdot Trace\left[F \cdot F^t\right] = (z + 1 - 2x^2)^2 + \left(\frac{y}{2} + yz\right)^2 + \left(\frac{3x}{2} + xz\right)^2,$$

where the electromagnetic-like matrix is $F = -N$.

Remark 1.4. *For more details about Lagrangian Yang-Mills electromagnetic-like energy, see the works* [6] *and* [1].

It is known that the *matrix of deviation curvature* from the Kosambi-Cartan-Chern (KCC) geometrical theory is given by the formula [3]

$$P = \left(P^i_j\right)_{i,j=\overline{1,3}} = \frac{\partial N}{\partial x^k} y^k + \left(\frac{\delta \mathcal{E}^i}{\delta x^j}\right)_{i,j=\overline{1,3}},$$

where

$$\mathcal{E}^i = 2G^i - N^i_j y^j = -\frac{1}{2}\left(\frac{\partial X^i}{\partial x^j} - \frac{\partial X^j}{\partial x^i}\right) y^j - \frac{\partial X^j}{\partial x^i} X^j$$

is the *first invariant of the semispray* of the Lagrangian (3). Using direct computations, we obtain that we have (Neagu and Ovsiyuk [9]):

$$\frac{\delta \mathcal{E}^i}{\delta x^j} = -\frac{1}{2}\left(\frac{\partial^2 X^i}{\partial x^j \partial x^k} - \frac{\partial^2 X^k}{\partial x^i \partial x^j}\right) y^k - \frac{\partial^2 X^k}{\partial x^i \partial x^j} X^k - \frac{\partial X^k}{\partial x^i}\frac{\partial X^k}{\partial x^j} -$$
$$-\frac{1}{4}\left(\frac{\partial X^k}{\partial x^j} - \frac{\partial X^j}{\partial x^k}\right)\left(\frac{\partial X^i}{\partial x^k} - \frac{\partial X^k}{\partial x^i}\right),$$

where $i, j \in \{1, 2, 3\}$.

Following the geometrical ideas from Böhmer et al. [2], we recall that the Jacobi stability or instability has the geometrical meaning that the trajectories of the Euler-Lagrange equations (4) are bunching together or are dispersing (or even are chaotic). Moreover, using also the ideas from Bucătaru and Miron [3], it follows that the behavior of the neighboring solutions of the Euler-Lagrange equations (4) is *Jacobi stable* if and only if the real parts of the eigenvalues of the deviation tensor P are strictly negative everywhere, and *Jacobi unstable*, otherwise (Böhmer et al. [2]).

1.2. **The Hamilton geometry produced by the Rabinovich-Fabrikant dynamical system**

We construct the *least squares Hamiltonian* $H : T^*M \to \mathbb{R}$ related with the Lagrangian (3) by taking

$$\begin{gathered} H(x,p) = \frac{\delta^{ij}}{4} p_i p_j + X^k(x) p_k \Leftrightarrow \\ H(x,p) = \frac{1}{4}\left(p_1^2 + p_2^2 + p_3^2\right) + X^1(x)p_1 + X^2(x)p_2 + X^3(x)p_3, \end{gathered} \tag{5}$$

where $p_r = \partial L / \partial y^r$ and $H = p_r y^r - L$.

Via the Hamilton geometry on the cotangent bundle T^*M and the Hamiltonian least squares variational method for dynamical systems (see the monographs Miron et al. [7] and Neagu-Oană [8]), it follows that we can build a natural and distinct collection of nonzero Hamiltonian geometrical objects (such as nonlinear connection and d-torsions), which also characterize the Rabinovich-Fabrikant dynamical system (1). In conclusion, we get the result [10]:

Proposition 1.4. *The* ***Hamiltonian canonical nonlinear connection*** *on the cotangent bundle T^*M, produced by the Rabinovich-Fabrikant dynamical system (1), is described by the symmetric matrix:*

$$N = \begin{pmatrix} 4xy + 2\gamma & 4z - 2x^2 & y - 2yz \\ 4z - 2x^2 & 2\gamma & 3x - 2xz \\ y - 2yz & 3x - 2xz & -4(\nu + xy) \end{pmatrix}.$$

Proof. The components of the Hamiltonian nonlinear connection on the cotangent bundle T^*M are (see [7])

$$N_{ij} = \frac{\partial^2 H}{\partial x^j \partial p_i} + \frac{\partial^2 H}{\partial x^i \partial p_j}.$$

By direct calculations, we find that we have $\mathbf{N} = J + J^t$. For more information, see the book [8]. □

Remark 1.5. *Obeying a change of coordinates $\tilde{x}^i = \tilde{x}^i(x^j)$, we recall that the components $N_{ij}(x,p)$ of the* ***Hamiltonian nonlinear connection*** *transform* *on T^*M by the rules* [7], [8]*:*

$$\tilde{N}_{jr} = N_{ki}\frac{\partial x^k}{\partial \tilde{x}^j}\frac{\partial x^i}{\partial \tilde{x}^r} - \frac{\partial x^i}{\partial \tilde{x}^r}\frac{\partial \tilde{p}_j}{\partial x^i}.$$

Moreover, the canonical Cartan linear connection on T^*M produced by the least squares Hamiltonian (5) has all adapted components equal to zero and the corresponding general formulas which define the Hamiltonian d-torsions are given by (see [7])

$$\mathbf{R}_k = \left(R_{kij} := \frac{\delta N_{ki}}{\delta x^j} - \frac{\delta N_{kj}}{\delta x^i}\right)_{i,j=\overline{1,3}},$$

where $\delta/\delta x^j = \partial/\partial x^j - N_{rj}\partial/\partial p_r$.

Remark 1.6. *It is a well known fact that, with respect to a change of coordinates $\tilde{x}^i = \tilde{x}^i(x^j)$, a Hamiltonian **d-tensor** $T^{ij\ldots}_{kl\ldots}(x,p)$ on the cotangent bundle T^*M obey the rules of transformation* [7], [8]*:*

$$T^{ij\ldots}_{kl\ldots} = \tilde{T}^{pm\ldots}_{rs\ldots}\frac{\partial x^i}{\partial \tilde{x}^p}\frac{\partial x^j}{\partial \tilde{x}^m}\frac{\partial \tilde{x}^r}{\partial x^k}\frac{\partial \tilde{x}^s}{\partial x^l}\cdots .$$

Consequently, we infer (for details, see also [8])

Proposition 1.5. *The Hamiltonian Cartan linear connection, produced by the Rabinovich-Fabrikant dynamical system (1), is described by the **d-torsion skew-symmetric matrices**:* $\mathbf{R}_k = \partial[J - J^t]/\partial x^k = -2R_k$, $\forall\, k = \overline{1,3}$.

2. Constant level Yang-Mills surfaces and illustrative plots

Surfaces of constant level are generally sets of points where a multiple variable function maintains a constant value. In this context, from our new geometric-physical approach, the surfaces of constant level of the Lagrangian Yang-Mills electromagnetic-like energy produced by the Rabinovich-Fabrikant dynamical system (1), may be relevant for the underlying dynamics. The surface of constant level Σ_ρ is expressed by the equation [10]

$$\Sigma_\rho : (z + 1 - 2x^2)^2 + \left(\frac{y}{2} + yz\right)^2 + \left(\frac{3x}{2} + xz\right)^2 = \rho^2 > 0.$$

Remark 2.1. *Assuming that $\mathcal{EYM} = 0$, the Rabinovich-Fabrikant dynamical system has no solutions. This is because the algebraic solutions of the system* *of equations $z + 1 - 2x^2 = 0$, $y/2 + yz = 0$, $3x/2 + xz = 0$ do not verify the initial Rabinovich-Fabrikant dynamical system.*

We present below several illustrative graphics of constant level Yang-Mills (YM-) surfaces associated with the Rabinovich-Fabrikant dynamical system.

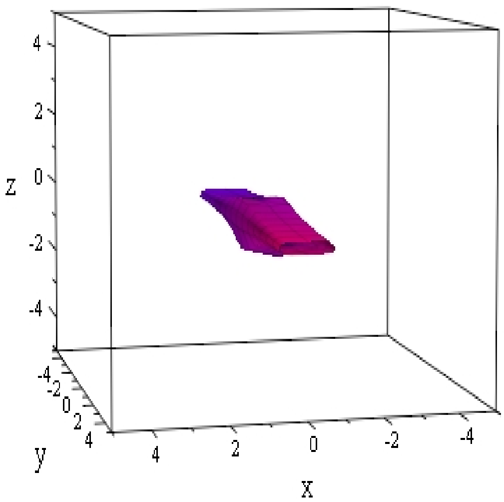

Fig.1: *The YM-surface for $\rho = 1$.*

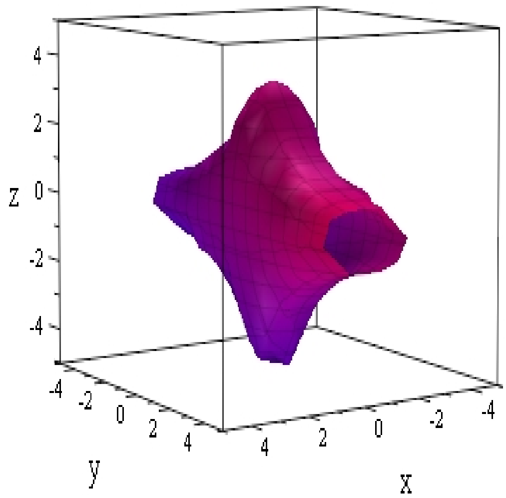

Fig.2: *The YM-surface for $\rho = 2\sqrt{5}$.*

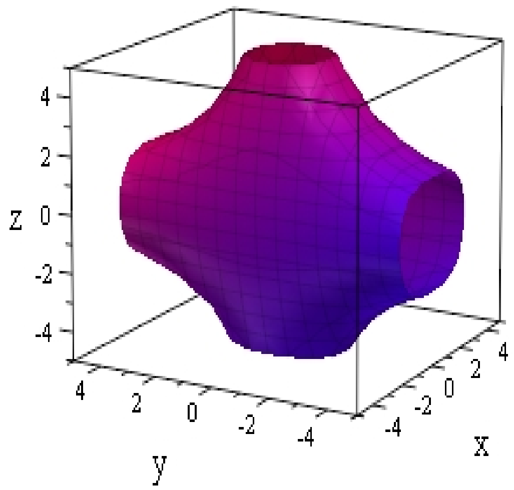

Fig.3: *The YM-surface for $\rho = 10$.*

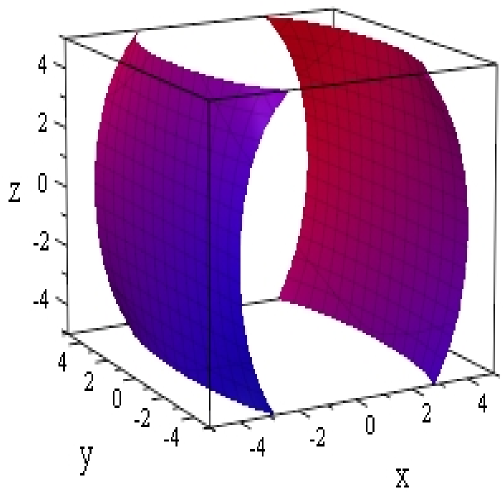

Fig.4: *The YM-surface for $\rho = 10\sqrt{10}$.*

Remark 2.2. *It is obvious that the above Yang-Mills surfaces are similar figures. The diameters of the tubes and the hole inside them increase with the increasing of the constant ρ. The shape of these surfaces, together with their* *natural parametrizations, can produce changes of variables that lead to solutions for the Rabinovich-Fabrikant system. However, this is an open problem of our geometrical investigation for the engineers specialized in the study of disturbances occurring in bridges and aircraft wings.*

3. **Parametrization of Yang-Mills surfaces. The first fundamental form. Geodesics**

We further derive the parametrization of the surface Σ_ρ. This constant level Yang-Mills surface can be parametrized by solving the system

$$\begin{cases} z+1-2x^2 = \rho\cos u\cos v \\ \dfrac{y}{2}+yz = \rho\cos u\sin v \\ \dfrac{3x}{2}+xz = \rho\sin u, \end{cases}$$

where $u \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, $v \in [0, 2\pi]$. In such a perspective, we arrive to the polynomial equation of degree three

$$x^3+px+q=0,$$

where

$$p = \frac{2\rho\cos u\cos v+1}{4}, \quad q = -\frac{\rho\sin u}{2}.$$

The solution of the above equation is given by the *Tartaglia-Cardano's formula*

$$x = \sqrt[3]{-\frac{q}{2}+\sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2}-\sqrt{\Delta}},$$

where

$$\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3.$$

We find that we have

$$\begin{cases} x = x(u,v) = \sqrt[3]{\dfrac{\rho\sin u}{4}+\dfrac{w}{24}} + \sqrt[3]{\dfrac{\rho\sin u}{4}-\dfrac{w}{24}}, \\ y = y(u,v) = \dfrac{2\rho\cos u\sin v}{1+2(\rho\cos u\cos v-1+2x^2)}, \\ z = z(u,v) = \rho\cos u\cos v - 1 + 2x^2, \end{cases} \tag{6}$$

where

$$w = \sqrt{\frac{108\rho^2\sin^2 u + (2\rho\cos u\cos v+1)^3}{3}}.$$

Through a laborious calculation, we find that

$$x_u = \frac{\partial x}{\partial u} = \frac{\rho}{12\sqrt[3]{\left[\dfrac{\rho \sin u}{4} + \dfrac{w}{24}\right]^2}} \cdot$$

$$\cdot \left\{ \cos u + \frac{[36\rho \cos u - (2\rho \cos u \cos v + 1) \cos v]^2 \sin u}{6w} \right\} +$$

$$+ \frac{\rho}{12\sqrt[3]{\left[\dfrac{\rho \sin u}{4} - \dfrac{w}{24}\right]^2}} \cdot$$

$$\cdot \left\{ \cos u - \frac{[36\rho \cos u - (2\rho \cos u \cos v + 1) \cos v]^2 \sin u}{6w} \right\},$$

$$x_v = \frac{\partial x}{\partial v} = \frac{-\rho(2\rho \cos u \cos v + 1)^2 \cos u \sin v}{72w\sqrt[3]{\left[\dfrac{\rho \cos u}{4} + \dfrac{w}{24}\right]^2}} + \frac{\rho(2\rho \cos u \cos v + 1)^2 \cos u \sin v}{72w\sqrt[3]{\left[\dfrac{\rho \cos u}{4} - \dfrac{w}{24}\right]^2}}.$$

Consequently, we obtain

$$y_u = \frac{\partial y}{\partial u} = \frac{-2\rho \sin v}{[1 + 2(\rho \cos u \cos v - 1 + 2x^2)]^2} \cdot$$

$$\cdot \left\{ [1 + 2(\rho \cos u \cos v - 1 + 2x^2)] \sin u + 2(-\rho \sin u \cos v + 4x x_u) \cos u \right\},$$

$$y_v = \frac{\partial y}{\partial v} = \frac{2\rho \cos u}{[1 + 2(\rho \cos u \cos v - 1 + 2x^2)]^2} \cdot$$

$$\cdot \left\{ [1 + 2(\rho \cos u \cos v - 1 + 2x^2)] \cos v + 2(\rho \cos u \sin v - 4x x_v) \sin v \right\},$$

$$z_u = \frac{\partial z}{\partial u} = -\rho \sin u \cos v + 4x x_u, \quad z_v = \frac{\partial z}{\partial v} = -\rho \cos u \sin v + 4x x_v.$$

In conclusion, we have the following parametrization for the Yang-Mills surface

$$\Sigma_\rho = \mathrm{Im}\ r, \ r(u, v) = (x(u, v), y(u, v), z(u, v)),$$

where $u \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, $v \in [0, 2\pi]$, and the functions $x(u, v), y(u, v), z(u, v)$ are given by the formulas (6).

It follows that *the first fundamental form* of the surface Σ_ρ is defined as the symmetric matrix [5]

$$g = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

Here we have $E = x_u^2 + y_u^2 + z_u^2,\ F = x_u x_v + y_u y_v + z_u z_v,\ G = x_v^2 + y_v^2 + z_v^2$, where $x_u,\ x_v,\ y_u,\ y_v,\ z_u,\ z_v$ are described above.

We recall that the equation of *geodesics* $C : v = v(u)$ of the Yang-Mills surface Σ_ρ is expressed as [5]:

$$\det\left[r_u + r_v\frac{dv}{du}, r_{uu} + 2r_{uv}\frac{dv}{du} + r_{vv}\left(\frac{dv}{du}\right)^2 + r_v\frac{d^2v}{du^2}, r_u \times r_v\right] = 0,$$

where

$$r_u = (x_u, y_u, z_u), \quad r_v = (x_v, y_v, z_v)$$

$$r_{uu} = (x_{uu}, y_{uu}, z_{uu}), \quad r_{uv} = (x_{uv}, y_{uv}, z_{uv}), \quad r_{vv} = (x_{vv}, y_{vv}, z_{vv}).$$

Open problem. Providing the complete family of solutions for the geodesic equation is a matter of further investigation.

Remark 3.1. *However, it is important to note that the shape of the constant level Yang-Mills surfaces, produced by the Rabinovich-Fabrikant dynamical system, may provide significant insights in the study of perturbations of bridges or aircraft wings.*

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