GLOBAL EXISTENCE AND NONEXISTENCE FOR A GENERALIZED PSEUDO-PARABOLIC EQUATION WITH VARIABLE EXPONENTS AND LOGARITHMIC NONLINEARITIES

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In this paper, we study pseudo-parabolic equations with logarithmic nonlinearity and a singular potential, involving variable exponents under suitable assumptions. Using energy methods, we show that solutions with negative initial energy blow up in finite time, and we then extend this blow-up result to a class of solutions with positive initial energy. Moreover, upper and lower estimates for the blow-up time are obtained. In addition, we establish two global existence theorems under different assumptions on the initial data.

Keywords: pseudo-parabolic equation, singular potential, variable exponent, logarithmic nonlinearity, blow-up result, global Existence

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1. Introduction

Consider a bounded domain $\Omega \subset \mathbb{R}^n$, with a sufficiently smooth boundary $\partial\Omega$. The focus here is an initial-boundary value problem involving variable exponents, formulated as follows:

$$\begin{cases} |x|^{-s(x)}u_t - \Delta_{r(x)}u - a\Delta_{m(x)}u_t = bu|u|^{p(x)-2}\ln|u|, & \text{in } \Omega\times(0,T) \\ u(x,t) = 0, & \text{on } \partial\Omega\times(0,T) \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{1}$$

where the parameters $a, b > 0$ and the exponent functions $r(\cdot)$, $m(\cdot)$ and $p(\cdot)$ are given measurable functions on Ω satisfying

$$2 \le \varphi_1 \le \varphi(x) \le \varphi_2 < +\infty, \tag{2}$$

with

$$\varphi_1 := \operatorname{ess\,inf}_{x\in\Omega}\varphi(x), \quad \varphi_2 := \operatorname{ess\,sup}_{x\in\Omega}\varphi(x),$$

and the log-Hölder continuity condition:

$$|\varphi(x)-\varphi(y)| \le -\frac{A}{\log|x-y|}, \text{ for a.e. } x, y \in \Omega, \text{ with } |x-y| < \delta < 1,\ A > 0, \tag{3}$$

while the singularity exponent $s(\cdot)$ is taken to be continuous, positive, and satisfies

$$s(x) < \frac{n(p_1-2)}{p_2}, \text{ for all } x \in B(0,\eta) \subset \Omega,\ \eta > 0. \tag{4}$$

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The nonlinear term $\Delta_{q(x)}u = \operatorname{div}(|\nabla u|^{q(x)-2}\nabla u)$ is called $q(x)$–Laplacian and $|x| = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2}$.

Pseudo-parabolic type equations have emerged in various branches of physics and engineering as effective models characterized by mixed time and space derivatives appearing in the highest order term. Such equations arise in semiconductor physics [2, 17, 18], the unidirectional propagation of nonlinear dispersive long waves [7], fluid flow in fissured porous media [5, 6], two phase flow in porous media with dynamical capillary pressure [16, 23], heat conduction in two-temperature systems [10], and population aggregation phenomena [24]. Rigorous mathematical analysis of such systems began with the works [27, 26], which established foundational results regarding their well-posedness and regularity. Extensive research has since developed, yielding a broad class of results on the existence, uniqueness, and qualitative behavior of solutions.

Recent years have seen increasing attention devoted to equations featuring logarithmic nonlinearity (see [22, 8]). In this context, the authors of [15] undertook a study of the pseudo-parabolic p-Laplacian equation with a logarithmic source term, subject to homogeneous Dirichlet boundary conditions:

$$u_t - \Delta u_t - \Delta_p u = |u|^{q-2}u\log(|u|),\ (x,t) \in \Omega \times (0,T),$$

with the constraints $2 < p < q < p(1 + 2/n)$. Their analysis established the existence and uniqueness of local solutions, obtained results on global existence and asymptotic behavior, and finally proved a finite-time blow-up theorem. Ding et al.[14] established global existence and blow-up results for the above equation with more general exponents. These findings, like the prior ones, were obtained via the potential well method and concavity method.

Variable-exponent equations provide an effective approach to modeling systems and materials with spatially dependent, non-uniform properties. Representative examples include electrorheological fluids, fluids with temperature-dependent viscosity, nonlinear viscoelasticity, porous media filtration, and image processing (see [1, 20, 3, 11]). These equations are commonly referred to as models with nonstandard growth conditions [4]. In the study conducted in [12], a pseudo-parabolic equation featuring variable-exponent nonlinearities was analyzed:

$$u_t - \nu\Delta u_t - \operatorname{div}\left(|\nabla u|^{m(x)-2}\nabla u\right) = |u|^{p(x)-2}u, \quad \text{in } \Omega \times (0,T),$$

subject to initial and Dirichlet boundary conditions. Employing a differential inequality approach, the authors derived upper and lower bounds for the blow-up time, contingent upon specific conditions on the variable exponents $p(\cdot)$, $m(\cdot)$, and characteristics of the initial state. Extending this framework, [21] employed the concavity method together with refined inequality estimates to further characterize blow-up phenomena in the case of positive initial energy. More recently, [29] expanded these results, showing that blow-up can occur even for arbitrarily large initial energy.

Problems featuring singular potentials play a significant role in modeling compressible fluid flow within homogeneous, isotropic, and rigid porous media (see

[30, 28]). The authors in [19] focused on the pseudo-parabolic equation containing a singular potential

$$\frac{u_t}{|x|^s} - \Delta u_t - \Delta u = |u|^{p-2}u, \quad (x,t) \in \Omega \times (0,T)$$

with $0 \leq s \leq 2$, $2 < p < \frac{2n}{n-2}$, $n \geq 3$. Their investigation yielded significant results, including the establishment of global existence, detailed descriptions of asymptotic solution behavior, and conditions under which finite-time blow-up occurs. In [25], Pişkin et al. study a higher-order reaction–diffusion parabolic problem with a singular potential and a time-dependent source coefficient. They consider the initial–boundary value problem

$$\begin{cases} \dfrac{z_t}{|x|^{2m}} + \mathcal{A}z = \alpha(t)|z|^{r-1}z, & (x,t) \in \Omega \times (0,T), \\ \dfrac{\partial^i z}{\partial \nu^i} = 0, \quad i = 0,1,\dots,m-1, & (x,t) \in \partial\Omega \times (0,T), \\ z(x,0) = z_0(x) \in H_0^m(\Omega) \cap L^{r+1}(\Omega), & x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, $\mathcal{A} = (-\Delta)^m$ with integer $m > 1$, $r > 1$, and $\alpha \in C^1[0,\infty)$ satisfies $\alpha(0) > 0$ and $\alpha'(t) \geq 0$ for all $t \geq 0$. Using concavity method, they first establish finite-time blow-up of solutions under suitable conditions and derive both lower and upper bounds for the blow-up time; moreover, they prove the existence of a global weak solution in an appropriate stable set of initial data.

The primary aim of this study is to investigate the blow-up behavior and global existence of solutions to Problem (1) for different cases of the initial energy. Specifically, sufficient conditions are derived on the variable exponent functions that guarantee finite-time blow-up, together with a rigorous upper bound for the blow-up time obtained via differential inequality techniques. Furthermore, two theorems are stated and proved under distinct assumptions, providing conditions that ensure the global existence of solutions.

2. Preliminaries

In this section, we present some preliminary information and fundamental results about Lebesgue and Sobolev spaces with variable exponents. Let $p : \Omega \subset \mathbb{R}^n \to [1,\infty]$ be a measurable function. The space $L^{p(\cdot)}(\Omega)$ consists of all real measurable functions $u : \Omega \to \mathbb{R}$ for which there exists $\lambda > 0$ such that

$$\varrho_{p(\cdot)}(\lambda u) := \int_\Omega |\lambda u(x)|^{p(x)}\, dx < \infty.$$

The variable-exponent Lebesgue space equipped with the Luxemburg-type norm

$$\|u\|_{p(\cdot)} := \inf\left\{\lambda > 0 : \varrho_{p(.)}(u/\lambda) \leq 1\right\},$$

is a Banach space. The Banach space $W^{k,p(\cdot)}(\Omega)$ ($k \in \mathbb{N}$), known as the variable-exponent Sobolev space, is defined as follows

$$\begin{cases} W^{k,p(.)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) : \ |D^\alpha u| \in L^{p(\cdot)}(\Omega) \text{ with } |\alpha| \leq k\right\}, \\ \|u\|_{W^{k,p(\cdot)}(\Omega)} = \sum_{0 \leq |\alpha| \leq k} \|D^\alpha u\|_{p(.)}. \end{cases}$$

Furthermore, we set $W_0^{k,p(\cdot)}(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in $W^{k,p(\cdot)}(\Omega)$. Similarly to the classical Sobolev spaces, we define $W^{-k,p'(\cdot)}(\Omega)$ as the dual of $W_0^{k,p(\cdot)}(\Omega)$, where $1/p(\cdot) + 1/p'(\cdot) = 1$. For more details on function spaces with variable exponents, we refer the reader to the monograph [13].

Lemma 1. *([13]) Let $p, q \geq 1$ be measurable functions defined on Ω such that $p(x) \leq q(x)$ for almost every $x \in \Omega$, then $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous.*

Lemma 2. *([13]) If $p(\cdot)$ is a measurable function on Ω satisfying (2), then for any $u \in L^{p(\cdot)}(\Omega)$ we have*

$$\min\left\{\|u\|_{p(\cdot)}^{p_1}, \|u\|_{p(\cdot)}^{p_2}\right\} \leq \varrho_{p(\cdot)}(u) \leq \max\left\{\|u\|_{p(\cdot)}^{p_1}, \|u\|_{p(\cdot)}^{p_2}\right\}.$$

Lemma 3. *(Young's inequality [13]). Let $h, i, j \geq 1$ be measurable functions defined on Ω such that*

$$\frac{1}{h(y)} = \frac{1}{i(y)} + \frac{1}{j(y)}, \text{ for a.e. } y \in \Omega.$$

Then for all $a, b \geq 0$,

$$\frac{(ab)^{h(.)}}{h(.)} \leq \frac{(a)^{i(.)}}{i(.)} + \frac{(b)^{j(.)}}{j(.)}.$$

Lemma 4. *(Hölder's Inequality [13]). Let $h, i, j \geq 1$ be measurable functions defined on Ω such that*

$$\frac{1}{h(y)} = \frac{1}{i(y)} + \frac{1}{j(y)}, \text{ for a.e. } y \in \Omega.$$

If $f \in L^{i(\cdot)}(\Omega)$ and $g \in L^{j(\cdot)}(\Omega)$, then $fg \in L^{h(\cdot)}(\Omega)$ and

$$\|fg\|_h \leq 2\|f\|_i \|g\|_j.$$

Lemma 5. *(Poincaré's inequality [13]). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $p(\cdot)$ satisfies (3). Then*

$$\|u\|_{p(\cdot)} \leq C\|\nabla u\|_{p(\cdot)}, \quad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega),$$

where the positive constant depends on Ω, p_1, and p_2. Consequently, the space $W_0^{1,p(\cdot)}(\Omega)$ has an equivalent norm given by $\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{p(\cdot)}$.

Lemma 6. *(Sobolev embedding theorem [13]). Let $p(\cdot) \in C(\overline{\Omega})$, and assume that $\operatorname{essinf}_{x \in \overline{\Omega}}\big(p^*(x) - q(x)\big) > 0$ holds, with*

$$p(x) \leq q(x) \text{ and } p^*(x) = \begin{cases} \dfrac{np(x)}{n - p(x)}, & \text{if } p(x) < n, \\ +\infty, & \text{if } p(x) \geq n. \end{cases}$$

Then the Sobolev embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.

Lemma 7. *We define the function*

$$h(\alpha) = \frac{1}{r_2} \min\left\{\alpha^{r_1/r_2}, \alpha\right\} - \frac{b}{p_1 e \sigma} B_1^{p_2+\sigma} \alpha^{(p_2+\sigma)/r_2}, \, \alpha \geq 0,$$

where $B_1 > 0$ and $\sigma > 0$ are constants. Then:

- $h(\alpha) \in C\big([0, +\infty)\big)$.

- $\lim\limits_{\alpha\to+\infty} h(\alpha) = -\infty$ *and* $h(0) = 0$.
- *There exists a unique* $\alpha^* > 0$ *such that,* h *is strictly increasing for* $0 \le \alpha < \alpha^*$*, and strictly decreasing for* $\alpha \ge \alpha^*$*, where*

$$\alpha^* = \begin{cases} \left[\dfrac{p_1 e\sigma}{b(p_2+\sigma)B_1^{p_2+\sigma}}\right]^{\frac{r_2}{p_2+\sigma-r_2}}, & \text{if } b > \dfrac{p_1 e\sigma}{(p_2+\sigma)B_1^{p_2+\sigma}}, \\ \left[\dfrac{r_1 p_1 e\sigma}{b r_2(p_2+\sigma)B_1^{p_2+\sigma}}\right]^{\frac{r_2}{p_2+\sigma-r_1}}, & \text{if } 0 < b < \dfrac{r_1 p_1 e\sigma}{r_2(p_2+\sigma)B_1^{p_2+\sigma}}, \\ 1, & \text{if } \dfrac{r_1 p_1 e\sigma}{r_2(p_2+\sigma)B_1^{p_2+\sigma}} \le b \le \dfrac{p_1 e\sigma}{(p_2+\sigma)B_1^{p_2+\sigma}}, \end{cases}$$

and

$$h(\alpha^*) = \begin{cases} \left(\frac{1}{r_2} - \frac{1}{p_2+\sigma}\right)\alpha^*, & \text{if } b > \dfrac{p_1 e\sigma}{(p_2+\sigma)B_1^{p_2+\sigma}}, \\ \left(\dfrac{1}{r_2} - \dfrac{r_1}{r_2(p_2+\sigma)}\right)(\alpha^*)^{\frac{r_1}{r_2}}, & \text{if } 0 < b < \dfrac{r_1 p_1 e\sigma}{r_2(p_2+\sigma)B_1^{p_2+\sigma}}, \\ \frac{1}{r_2} - \frac{b}{p_1 e\sigma}B_1^{p_2+\sigma}, & \text{if } \dfrac{r_1 p_1 e\sigma}{r_2(p_2+\sigma)B_1^{p_2+\sigma}} \le b \le \dfrac{p_1 e\sigma}{(p_2+\sigma)B_1^{p_2+\sigma}}. \end{cases}$$

Proof. **(i) Continuity and boundary behavior.** h is a composition of continuous functions on $[0,+\infty)$, so $h \in C([0,+\infty))$. Clearly $h(0) = 0$. Since $(p_2+\sigma)/r_2 > 1$ (because $p_2 > r_2$ by assumption (8)), the term $-\frac{b}{p_1 e\sigma}B_1^{p_2+\sigma}\alpha^{(p_2+\sigma)/r_2}$ dominates as $\alpha \to +\infty$, so $h(\alpha) \to -\infty$.

(ii) Existence and uniqueness of the maximum α^*. We differentiate h on each region. For $\alpha \ge 1$: $\min\{\alpha^{r_1/r_2}, \alpha\} = \alpha$, so

$$h'(\alpha) = \frac{1}{r_2} - \frac{b(p_2+\sigma)}{p_1 e\sigma r_2} B_1^{p_2+\sigma}\, \alpha^{(p_2+\sigma-r_2)/r_2}.$$

Setting $h'(\alpha^*) = 0$ yields

$$\alpha^* = \left(\frac{p_1 e\sigma}{b(p_2+\sigma)B_1^{p_2+\sigma}}\right)^{r_2/(p_2+\sigma-r_2)}, \quad \text{valid when } b > \frac{p_1 e\sigma}{(p_2+\sigma)B_1^{p_2+\sigma}}.$$

For $0 < \alpha < 1$: $\min\{\alpha^{r_1/r_2}, \alpha\} = \alpha^{r_1/r_2}$, so

$$h'(\alpha) = \frac{r_1}{r_2^2}\,\alpha^{(r_1-r_2)/r_2} - \frac{b(p_2+\sigma)}{p_1 e\sigma r_2} B_1^{p_2+\sigma}\, \alpha^{(p_2+\sigma-r_2)/r_2}.$$

Setting $h'(\alpha^*) = 0$ yields

$$\alpha^* = \left(\frac{r_1 p_1 e\sigma}{r_2 b(p_2+\sigma)B_1^{p_2+\sigma}}\right)^{r_2/(p_2+\sigma-r_1)}, \quad \text{valid when } b < \frac{r_1 p_1 e\sigma}{r_2(p_2+\sigma)B_1^{p_2+\sigma}}.$$

In the intermediate range of b, $\alpha^* = 1$. In all cases $h'(\alpha^*) = 0$, $h'(\alpha) > 0$ for $\alpha \in (0, \alpha^*)$, and $h'(\alpha) < 0$ for $\alpha > \alpha^*$, confirming that α^* is the unique global maximum. The explicit formulas for $h(\alpha^*)$ follow by direct substitution, giving the three cases stated in the lemma. □

Next, we state the well-posedness theorem for our problem 1.

Proposition 1. *Let* $r(\cdot), m(\cdot), p(\cdot)$ *satisfy conditions* (2) *and* (3)*. In addition, assume that*

$$essinf_{x\in\overline{\Omega}}\big(r^*(x)-p(x)\big)>0,$$

with

$$r_2<p_1 \text{ and } r^*(x)=\begin{cases}\dfrac{nr(x)}{n-r(x)}, & \text{if } r(x)<n,\\ +\infty, & \text{if } r(x)\geq n,\end{cases}$$

and $s(\cdot)$ *satisfies* (4)*. Then, for any given* $u_0\in W_0^{1,r(\cdot)}(\Omega)$*, the problem* (1) *admits a unique local weak solution such that*

$$u\in L^\infty((0,T),W_0^{1,r(\cdot)}(\Omega)),\ u_t\in W_0^{1,m(\cdot)}(\Omega\times(0,T)),\ u_t/|x|^{\frac{s(\cdot)}{2}}\in L^2(\Omega\times(0,T)). \quad (5)$$

The proof of this proposition can be established employing the Galerkin method as in the work [19]. The modified energy functional $E(t)$ associated with the problem (1), defined as

$$E(t)=\int_\Omega\frac{1}{r(x)}|\nabla u|^{r(x)}\mathrm{d}x+\int_\Omega\frac{b}{p^2(x)}|u|^{p(x)}\mathrm{d}x-\int_\Omega\frac{b}{p(x)}|u|^{p(x)}\ln|u|\mathrm{d}x. \quad (6)$$

Multiplying equation (1) by u_t and integrating over Ω, we easily obtain

$$E'(t)=-\int_\Omega\left[\frac{|u_t|^2}{|x|^{s(x)}}+a|\nabla u_t|^{m(x)}\right]\mathrm{d}x\leq 0,\text{ a.e } t\in[0,T).$$

3. Blow-up results

In this section, we prove that the solution (5) blows up in finite time whenever

$$2\leq m_1\leq m(x)\leq m_2\leq r_1\leq r(x)\leq r_2<p_1\leq p(x)\leq p_2<r^*(x), \quad (7)$$

and the initial energy is negative. Then we extend this result to certain solutions with positive initial energy. Now we present some useful lemmas that will be used throughout this section.

Lemma 8. *The following well-known algebraic inequality holds:*

$$\lambda^\mu\leq\lambda+1\leq\left(1+\frac{1}{\varepsilon}\right)(\lambda+\varepsilon),\quad\forall\,\lambda\geq 0,\ 0<\mu\leq 1,\ \varepsilon>0. \quad (8)$$

Lemma 9. *Assume that* $p(\cdot)$ *is a measurable function on* Ω *satisfying* (2)*, and that* $s(\cdot)\in C(\overline{\Omega})$ *satisfies* (4)*. Then, for any* $u\in L^{p(\cdot)}(\Omega)$*, there exists* $B>0$ *such that*

$$\int_\Omega|x|^{-s(x)}|u|^2\mathrm{d}x\leq B\|u\|^2_{p(x)}.$$

Proof. Applying Hölder's inequality, we obtain

$$\int_\Omega|x|^{-s(x)}|u|^2\mathrm{d}x\leq\big\||x|^{-s(x)}\big\|_{\frac{p_2}{p_1-2}}\,\||u|^2\|_{\frac{p(x)}{2}}.$$

It is worth noting that, when the singularity function $s(\cdot)$ satisfies (4), then

$$0<B:=\big\||x|^{-s(x)}\big\|_{\frac{p_2}{p_1-2}}<+\infty,$$

and thus B is well defined. Recalling the definition of the Luxemburg-type norm in the Lebesgue space with variable exponent, we have

$$\||u|^2\|_{\frac{p(x)}{2}} = \|u\|^2_{p(x)},$$

we finally obtain

$$\int_\Omega |x|^{-s(x)}|u|^2 \mathrm{d}x \le B\,\|u\|^2_{p(x)}.$$

$\square$

Define $H(t) = -E(t)$. Then we obtain the following corollary.

Corollary 1. *Let u be the solution of* (1).*Then for every t in $[0,T)$, $H'(t) \ge 0$ and*

$$0 < H(0) \le H(t) \le \int_\Omega \frac{b}{p(x)}|u|^{p(x)} \ln|u| \ \mathrm{d}x.$$

The first result of this section is given in the following theorem.

Theorem 1. *Let the assumptions of Proposition 1 and condition* (7) *be fulfilled. Assume further that $E(0) < 0$. Then, the solution of problem* (1) *given by Proposition 1 blows up in finite time.*

Proof. We define the auxiliary function

$$L(t) := H^{1-\alpha}(t) + \frac{\varepsilon}{2}\int_\Omega |x|^{-s(x)}|u|^2 \mathrm{d}x, \tag{9}$$

where $\alpha = (r_1 - m_2)/r_2(m_2 - 1)$ and $\varepsilon > 0$ will be chosen later. Differentiating the auxiliary function $L(t)$ and using equation (1) together with (6) gives

$$\begin{aligned} L'(t) \ge &(1-\alpha)H^{1-\alpha}(t)H'(t) + \varepsilon\beta\left[H(t) + \varrho_{r(x)}(\nabla u) + \varrho_{p(x)}(u)\right] \\ &- a\varepsilon\int_\Omega \nabla u \nabla u_t |\nabla u_t|^{m(x)-2}\mathrm{d}x, \end{aligned} \tag{10}$$

where $\beta = \min\{(p_1/r_1) - 1, (bp_1/p_2^2)\} > 0$. Using Young's inequality, we estimate the last term in (10) as follows:

$$\begin{aligned} \int_\Omega |\nabla u_t|^{m(x)-1}|\nabla u|\,\mathrm{d}x \le &\frac{(m_2-1)k^2}{m_2}\int_\Omega \theta^{\frac{-m(x)}{m(x)-1}}|\nabla u_t|^{m(x)}\,\mathrm{d}x \\ &+ \frac{k^{-1}}{m_1}\int_\Omega |\nabla u|^{r(x)}\,\mathrm{d}x + \frac{1}{m_1}\int_\Omega \theta^{\frac{mr(x)}{(r-m)(x)}}\mathrm{d}x, \end{aligned}$$

which is valid for every $k > 1$ and $\theta(x,t) > 0$ for all $(x,t) \in \Omega \times \mathbb{R}$. Then by choosing θ so that

$$\theta^{-\frac{m(x)}{m(x)-1}} = k \cdot H^{-\alpha}(t),$$

and by using the wel-known algebraic inequality, there existe $C_1 > 0$

$$\int_\Omega |\nabla u_t|^{m(x)-1}|\nabla u|\,\mathrm{d}x \le \frac{C_1}{km_1}\left[\varrho_{r(x)}(\nabla u) + H(t)\right] + \frac{(m_2-1)\,k^3}{am_2}H^{-\alpha}H'(t). \tag{11}$$

When (10) and (11) are combined, it gives

$$\begin{aligned} L'(t) \geq & \left[(1-\alpha) - \frac{\varepsilon\,(m_2-1)\,k^3}{m_2}\right] H^{-\alpha}(t)\, H'(t) \\ & + \varepsilon\left(\beta - \frac{C_1}{km_1}\right)\left[H(t) + \varrho_{r(x)}(\nabla u) + \varrho_{p(x)}(u)\right]. \end{aligned} \tag{12}$$

Here, we select k sufficiently large such that

$$\gamma = \beta - \frac{C_1}{km_1} > 0.$$

After k has been fixed we determine ε small enough to ensure

$$(1-\alpha) - \frac{\varepsilon\,(m_2-1)\,k^3}{m_2} \geq 0.$$

Thus, (12) takes the form

$$L'(t) \geq \gamma\varepsilon\left[H(t) + \varrho_{r(x)}(\nabla u) + \varrho_{p(x)}(u)\right]. \tag{13}$$

Consequently, we have

$$L(t) \geq L(0) > 0, \text{ for all } t \geq 0.$$

On the other hand, from Lemma 9 and (8):

$$\begin{aligned} \left(\int_\Omega \frac{|u|^2}{|x|^{s(x)}}\mathrm{d}x\right)^{\frac{1}{1-\alpha}} & \leq \left[B\|u\|^2_{p(x)}\right]^{\frac{1}{1-\alpha}} \leq (B)^{\frac{1}{1-\alpha}}\, \|u\|^{\frac{2}{1-\alpha}}_{p(x)} \\ & \leq C_2\left[\min\left\{\|u\|^{p_1}_{p(x)}, \|u\|^{p_2}_{p(x)}\right\} + H(t)\right] \\ & \leq C_2\left[\varrho_{p(x)}(u) + H(t)\right], \end{aligned}$$

Then we see that

$$L^{\frac{1}{1-\alpha}}(t) \leq 2^{\frac{1}{1-\alpha}}\left[H(t) + \left[\int_\Omega \frac{|u|^2}{|x|^{s(x)}}\mathrm{d}x\right]^{\frac{1}{1-\alpha}}\right] \leq C_3\left[H(t) + \varrho_{p(x)}(u)\right]. \tag{14}$$

From (13) and (14), we conclude that

$$L'(t) \geq \Lambda_1 L^{\frac{1}{1-\alpha}}(t), \text{ for all } t \geq 0, \tag{15}$$

where Λ_1 is a positive constant depending on $\Omega, n, u_0, m_{1,2}$ and $p_{1,2}$ only. Therefore, $L(t)$ blows up in finite time

$$T_1^* \leq \frac{1-\alpha}{\Lambda_1\alpha\,[L(0)]^{\frac{\alpha}{1-\alpha}}}.$$

This completes the proof. □

At the end of this section, we establish a blow-up result for solutions with low-level positive initial energy. Before presenting this result, we first state a key lemma central to the argument.

Lemma 10. *Under the assumptions of Lemma 6 and the following conditions*

$$\|\nabla u_0\|_{r(x)}^{r_2} > \alpha^* > 0,\ E(0) < h(\alpha^*),$$

then there exists a constant $\alpha_2 > \alpha^*$ such that

$$\|\nabla u(\cdot,t)\|_{r(x)}^{r_2} \geq \alpha_2, \quad t \geq 0. \tag{16}$$

Proof. From the definition of $E(t)$, we have

$$E(t) \geq \frac{1}{r_2}\int_\Omega |\nabla u|^r dx - \frac{b}{p_1}\int_\Omega |u|^{p(x)} \ln |u| \mathrm{d}x.$$

Using the properties of the logarithmic function and Lemma 6, we obtain

$$\begin{aligned} E(t) &\geq \frac{1}{r_2}\int_\Omega |\nabla u|^r dx - \frac{b}{p_1}\left[\int_{\Omega_-} |u|^{p(x)} \ln |u| \mathrm{d}x + \int_{\Omega_+} |u|^{p(x)} \ln |u| \mathrm{d}x\right] \\ &\geq \frac{1}{r_2}\int_\Omega |\nabla u|^r dx - \frac{b}{p_1 e\sigma}\|u(\cdot,t)\|_{p_2+\sigma}^{p_2+\sigma} \\ &\geq \frac{1}{r_2}\int_\Omega |\nabla u|^r dx - \frac{b}{p_1 e\sigma} B_1^{p_2+\sigma}\|\nabla u(\cdot,t)\|_{r(x)}^{p_2+\sigma}, \end{aligned}$$

where $\Omega_- := \{x \in \Omega; |u| \leq 1\}$, $\Omega_+ := \{x \in \Omega; |u| > 1\}$, $\sigma > 0$ is a sufficiently small constant such that $p_2 + \sigma < r^*(x)$, and B_1 is the optimal constant of the Sobolev embedding $W_0^{1,r(\cdot)}(\Omega) \hookrightarrow L^{p_2+\sigma}(\Omega)$. By Lemma 2

$$E(t) \geq \frac{1}{r_2}\min\left\{\|\nabla u(\cdot,t)\|_{r(x)}^{r_1}, \|\nabla u(\cdot,t)\|_{r(x)}^{r_2}\right\} - \frac{b}{p_1 e\sigma} B_1^{p_2+\sigma}\|\nabla u(\cdot,t)\|_{r(x)}^{p_2+\sigma},$$

we put $\alpha = \|\nabla u(\cdot,t)\|_{r(x)}^{r_2}$, then

$$E(t) \geq \frac{1}{r_2}\min\left\{\alpha^{\frac{r_1}{r_2}}, \alpha\right\} - \frac{b}{p_1 e\sigma} B_1^{p_2+\sigma}\alpha^{\frac{p_2+\sigma}{r_2}} = h(\alpha), \quad tcge0. \tag{17}$$

Therefore, it follows that $h\left(\|\nabla u_0\|_{r(x)}^{r_2}\right) \leq E(0)$. Since $E(0) < h(\alpha^*)$, there exists a positive constant $\alpha_2 > \alpha^*$ such that $E(0) = h(\alpha_2)$. It implies that $\|\nabla u_0\|_{r(x)}^{r_2} \geq \alpha_2 > \alpha^*$. In order to verify (16), let us proceed by contradiction. Suppose that, for some $t_0 > 0$, $\|\nabla u(t_0)\|_{r(\cdot)}^{r_2} < \alpha_2$. Then there exists $t_1 > 0$ such that $\alpha^* < \|\nabla u(t_1)\|_{r(\cdot)}^{r_2} < \alpha_2$. Using the monotonicity of $h(\alpha)$ (Lemma 7), we obtain

$$E(t_1) > h\left(\|\nabla u(t_1)\|_{r(\cdot)}^{r_2}\right) > E(0),$$

which is in contradiction with $E(t) \leq E(0)$, for all $t \in (0,T)$. Hence, (16) is proved. □

We set $F(t) = h(\alpha^*) - E(t)$ and

$$V(t) := F^{1-\alpha}(t) + \frac{\varepsilon}{2}\int_\Omega |x|^{-s(x)}|u|^2 \mathrm{d}x,$$

where $\varepsilon > 0$ and $0 < \alpha < 1$ will be chosen as in the proof of Theorem 1. Assuming $E(0) < h(\alpha^*)$, it follows that $0 < F(0) \leq F(t)$ for all t in $[0,T)$. By appealing to (8), Lemma 9 and Lemma 16, and adapting the arguments from (9) through (15) in the proof of Theorem 1, we establish the following result.

Theorem 2. *Assume the hypotheses of Lemma 9 hold. Then the solution to problem* (1)*, as constructed in Proposition 1, exhibits finite-time blow-up.*

Remark 1. The proofs of Theorems 1 and 2 provide explicit upper bounds for the finite-time blow-up of their respective solutions:

$$T_1^* \leq \frac{1-\alpha}{\Lambda_1 \alpha \left[L(0)\right]^{\frac{\alpha}{1-\alpha}}},\ T_2^* \leq \frac{1-\alpha}{\Lambda_2 \alpha \left[V(0)\right]^{\frac{\alpha}{1-\alpha}}}.$$

4. **Global existence results**

In this section, we state and prove two theorems under different assumptions on the initial energy, the initial data, and a certain constant β.

Remark 2. Theorem 1 ensures that every global solution satisfies $E(0) \geq 0$; therefore, throughout this section we assume that $E(0) \geq 0$.

Lemma 11. *If the conditions of Lemma 6 and the following conditions hold*

$$\|\nabla u_0\|_{r(x)}^{r_2} < \alpha^*, 0 \leq E(0) < h(\alpha^*),$$

then there exists a constant $0 \leq \alpha_1 < \alpha^*$ *such that*

$$\|\nabla u(\cdot,t)\|_{r(x)}^{r_2} \leq \alpha_1, \quad t \geq 0, \tag{18}$$

and

$$E(t) \geq 0, \quad t \geq 0.$$

Proof. In view of (17), we deduce that $h\left(\|\nabla u_0\|_{r(x)}^{r_2}\right) \leq E(0)$. The assumption $0 \leq E(0) < h(\alpha)$ guarantees the existence of two positive constants $\alpha_2 > \alpha > \alpha_1$ satisfying $h(\alpha_1) = E(0) = h(\alpha_2)$. We conclude that $\|\nabla u_0\|_{r(x)}^{r_2} \leq \alpha_1 < \alpha^*$. To prove (18), suppose to the contrary that, for some $t_0 > 0$, $\|\nabla u(t_0)\|_{r(\cdot)}^{r_2} > \alpha_1$. Then there exists $t_1 > 0$ such that $\alpha_1 < \|\nabla u(t_1)\|_{r(\cdot)}^{r_2} < \alpha^*$. Due to the monotonicity of $h(\alpha)$ (Lemma 7), we deduce

$$E(0) \leq h\left(\|\nabla u(t_1)\|_{r(\cdot)}^{r_2}\right) \leq E(t_1),$$

which contradicts $E(t) \leq E(0)$, for all $t \in (0,T)$. Consequently, (18) is established. From inequality (18), we obtain $h(\alpha) \geq 0$, which implies $E(t) \geq 0$, for all $t \geq 0$. $\square$

Theorem 3. *Let the assumptions of Proposition 1 and Lemma 11 be satisfied. Then Problem* (1) *admits a unique global weak solution such that*

$$u \in L^\infty((0,\infty), W_0^{1,r(\cdot)}(\Omega)),\ u_t \in W_0^{1,m(\cdot)}(\Omega \times (0,\infty)),\ u_t/|x|^{\frac{s(\cdot)}{2}} \in L^2(\Omega \times (0,\infty)).$$

Proof. Using the Sobolev embedding theorem, we estimate

$$\begin{aligned}
&\int_\Omega \frac{b}{p(x)}|u|^{p(x)} \ln|u| \mathrm{d}x \leq \frac{b}{p_1 e\sigma} B_1^{p_2+\sigma} \|\nabla u(\cdot,t)\|_{r(x)}^{p_2+\sigma} \\
&\leq \frac{r_2 b B_1^{p_2+\sigma}}{p_1 e\sigma} \max\left\{\|\nabla u\|_{r(x)}^{p_2+\sigma-r_2}, \|\nabla u\|_{r(x)}^{p_2+\sigma-r_1}\right\} \left[\int_\Omega \frac{1}{r(x)}|\nabla u|^{r(x)} \mathrm{d}x\right] \\
&\leq \frac{r_2 b B_1^{p_2+\sigma}}{p_1 e\sigma} \max\left\{\|\nabla u\|_{r(x)}^{p_2+\sigma-r_2}, \|\nabla u\|_{r(x)}^{p_2+\sigma-r_1}\right\} \left[E(t) + \int_\Omega \frac{b}{p(x)}|u|^{p(x)} \ln|u| \mathrm{d}x\right],
\end{aligned} \tag{19}$$

and since $\|\nabla u(t)\|_{r(\cdot)}^{r_2} < \alpha^*$, for all $t \geq 0$ it follows that

$$\int_\Omega \frac{b}{p(x)}|u|^{p(x)} \ln |u| \mathrm{d}x \leq \frac{r_2 b}{p_1 e\sigma} B_1^{p_2+\sigma} \max\left\{\alpha^{*\frac{p_2+\sigma-r_2}{r_2}}, \alpha^{*\frac{p_2+\sigma-r_1}{r_2}}\right\} \cdot \left[E(t) + \int_\Omega \frac{b}{p(x)}|u|^{p(x)} \ln |u| \mathrm{d}x\right].$$

Recalling the definition of the constant α^*, we conclude that

$$\frac{r_2 b}{p_1 e\sigma} B_1^{p_2+\sigma} \max\left\{\alpha^{*\frac{p_2+\sigma-r_2}{r_2}}, \alpha^{*\frac{p_2+\sigma-r_1}{r_2}}\right\} = \begin{cases} \dfrac{r_2 e\sigma}{p_2+\sigma}, & \alpha^* < 1, \\ \dfrac{r_1 e\sigma}{p_2+\sigma}, & \alpha^* > 1, \\ \frac{r_2 b}{p_1 e\sigma} B_1^{p_2+\sigma}, & \alpha^* = 1, \end{cases}$$

selecting $\sigma \leq \frac{1}{e}$ ensures that

$$\frac{r_2 b}{p_1 e\sigma} B_1^{p_2+\sigma} \max\left\{\alpha^{*\frac{p_2+\sigma-r_2}{r_2}}, \alpha^{*\frac{p_2+\sigma-r_1}{r_2}}\right\} \leq \frac{r_2}{p_2+\sigma} < 1.$$

This shows that

$$\int_\Omega \frac{b}{p(x)}|u|^{p(x)} \ln |u| \mathrm{d}x \leq \frac{p_2+\sigma}{p_2+\sigma-r_2} E(t) \leq \frac{p_2+\sigma}{p_2+\sigma-r_2} E(0),$$

and

$$\begin{aligned} \int_\Omega \frac{1}{r(x)}|\nabla u|^{r(x)} \mathrm{d}x + \int_\Omega \frac{b}{p^2(x)}|u|^{p(x)} \mathrm{d}x &\leq \left(1 + \frac{p_2+\sigma}{p_2+\sigma-r_2}\right) E(t) \\ &\leq \left(1 + \frac{p_2+\sigma}{p_2+\sigma-r_2}\right) E(0). \end{aligned}$$

On the other hand, we have

$$E(t) + \int_0^t \int_\Omega \left[\frac{|u_t|^2}{|x|^{s(x)}} + a|\nabla u_t|^{m(x)}\right] \mathrm{d}x = E(0),$$

since the energy $E(t) \geq 0$, then

$$\int_0^{+\infty} \int_\Omega \left[\frac{|u_t|^2}{|x|^{s(x)}} + a|\nabla u_t|^{m(x)}\right] \mathrm{d}x \leq E(0).$$

□

Finally, we prove another result in different cases than before. To reach this, we define a new function

$$I(t) = \int_\Omega |\nabla u|^{r(x)} \mathrm{d}x + \int_\Omega \frac{bp_1}{p^2(x)}|u|^{p(x)} \mathrm{d}x - \int_\Omega b|u|^{p(x)} \ln |u| \mathrm{d}x,$$

which satisfies

$$E(t) \geq \frac{p_1 - r_2}{p_1 r_2} \int_\Omega |\nabla u|^{r(x)} \mathrm{d}x + \frac{1}{p_1} I(t). \tag{20}$$

We now state and prove the following lemma, which plays a crucial role in the proof of Theorem 4.

Lemma 12. *Supposing that* $I(0) > 0$ *and*

$$\beta := \frac{b}{e\sigma} B_1^{p_2+\sigma} \max\left\{ \left(\frac{p_1 r_2}{p_1 - r_2} E(0)\right)^{\frac{p_2+\sigma-r_2}{r_2}}, \left(\frac{p_1 r_2}{p_1 - r_2} E(0)\right)^{\frac{p_2+\sigma-r_1}{r_1}} \right\} < 1,$$

then we obtain

$$I(t) > 0 \text{ for all } t \in [0, T).$$

Proof. Since this function is continuous on $[0, T)$, there exists $T_0 \in [0, T)$ such that

$$I(t) \geq 0 \text{ for all } t \in [0, T_0].$$

From inequality (20), we find that for all $t \in [0, T_0]$

$$E(t) \geq \frac{p_1 - r_2}{p_1 r_2} \int_\Omega |\nabla u|^{r(x)} \mathrm{d}x.$$

As $E(t)$ is decreasing, we obtain

$$\int_\Omega |\nabla u|^{r(x)} \mathrm{d}x \leq \frac{p_1 r_2}{p_1 - r_2} E(0).$$

In light of Lemma 2

$$\|\nabla u\|_{r(x)} \leq \max\left\{ \left(\frac{p_1 r_2}{p_1 - r_2} E(0)\right)^{\frac{1}{r_1}}, \left(\frac{p_1 r_2}{p_1 - r_2} E(0)\right)^{\frac{1}{r_2}} \right\}.$$

By combining this with inequality (19), we arrive at

$$\int_\Omega b|u|^{p(x)} \ln|u| \mathrm{d}x \leq \frac{b}{e\sigma} B_1^{p_2+\sigma} \max\left\{ \|\nabla u\|_{r(x)}^{p_2+\sigma-r_2}, \|\nabla u\|_{r(x)}^{p_2+\sigma-r_1} \right\} \left[\int_\Omega |\nabla u|^{r(x)} \mathrm{d}x\right]$$

$$\leq \frac{b}{e\sigma} B_1^{p_2+\sigma} \max\left\{ \left(\frac{p_1 r_2}{p_1 - r_2} E(0)\right)^{\frac{p_2+\sigma-r_2}{r_2}}, \left(\frac{p_1 r_2}{p_1 - r_2} E(0)\right)^{\frac{p_2+\sigma-r_1}{r_1}} \right\} \left[\int_\Omega |\nabla u|^{r(x)} \mathrm{d}x\right]$$

$$\leq \beta \left[\int_\Omega |\nabla u|^{r(x)} \mathrm{d}x + \int_\Omega \frac{b p_1}{p^2(x)} |u|^{p(x)} \mathrm{d}x\right],$$

for all $t \in [0, T_0]$, this guarantees that

$$I(t) > 0 \text{ for all } t \in [0, T_0].$$

Since we have $\beta < 1$ and $I(T_0) > 0$, we can repeat the preceding argument and thus extend T_0 to T. □

Theorem 4. *Let the assumptions of Lemma 12 be satisfied. Then the solution* (5) *of Problem* (1) *is global.*

Proof. Lemma 12 together with inequality (20) yields

$$\frac{p_1 - r_2}{p_1 r_2} \int_\Omega |\nabla u|^{r(x)} \mathrm{d}x \leq E(t),\ t \in [0, T).$$

By the monotonicity of $E(t)$

$$\int_\Omega |\nabla u|^{r(x)} \mathrm{d}x \leq \frac{p_1 r_2}{p_1 - r_2} E(0),\ t \in [0, T).$$

Proceeding as in the proof of Theorem 3, we establish that

$$\int_\Omega b|u|^{p(x)} \ln |u| \mathrm{d}x \le \frac{p_2\beta}{p_2-\beta} E(0),$$

and

$$\int_0^t \int_\Omega \left[\frac{|u_t|^2}{|x|^{s(x)}} + a|\nabla u_t|^{m(x)} \right] \mathrm{d}x \le E(0).$$

The proof is complete. □

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