ON ϕ-FIXED POINTS FOR IMPLICIT CONTRACTIVE CONDITIONS

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In this work, existence results of fixed points are proved for operators which fulfill weakly implicit contractive inequalities defined by means of auxiliary functions with adequate properties. Additionally, these fixed points are zeros of a function endowed with the lower semi continuity, in the setting of extended b-metric spaces.

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1. Introduction

Fixed point theory is a field in mathematics that appeared from the need to solve problems of the form $x = Tx$, where T is a self mapping. In this case, x is the fixed point of the contraction T. Problems which may be formulated by such equations arise often in Physics, Material Science, Chemistry, or Astronomy.

The so called Banach contraction principle by Cacciopoli [9] opened the way for researchers to conduct studies in this topical direction of mathematics. In this regard, we recall the work of Kannan [15], published in 1968, in which he introduced a class of weakly contractive mappings and generalized the classic fixed point theorem; this was extended further, with respect to the u-distance given by Ume [23]. Górnicki [11] proved fixed point theorems for Kannan-type mappings using different types of distances and contractive conditions. Zamfirescu [26] introduced a type of functions which entail the classic contractions, and those of Kannan and Chattrejea. Yen [25] used iterates of a certain order of the considered operator to design fixed point outcomes. Another interesting study is due to Suzuki [22], where a family of mappings which entails that of nonexpansive mappings is introduced. Kar and Veeramani [16] obtained fixed

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point results for a class of asymptotic nonexpansive mappings in a uniformly convex Banach space. In [7], Batra *et al.* enunciate fixed point theorems for a weaker class of F-Kannan mappings on a non complete metric space. The type of inequalities which feature the main operator provides another opportunity to extend further this direction. Wardowski [24] introduced a family of auxiliary functions which enabled the development of implicit conditions which ensure the existence of a fixed point. Khojasteh *et al.* [17] introduced the use of simulation functions in the fixed point theory. Their idea was further generalized by de Hierro and Shahzad [12], to the R-contractions.

One of the ways of extending fixed point theory is related to the setting on which operators are defined. In this respect, Bakhtin [5], and Czerwik [10] made a generalization of the classic metric space by adding a constant greater that one in the right part of the triangle inequality and so creating the b-metric space. This background has been used, for example, by Bota *et al.* [8], regarding the fixed point of a Ćirić type operator on a partially ordered b-metric space and applications to coupled fixed point theory. Kamran *et al.* [13] studied Feng and Liu type contractions in b-metric spaces which were used to solving integral equations. In their work, Barve *et al.* [6] presented a fixed point theorem for four pairs of mappings with respect with adequate properties in b-metric spaces. Volterra integral inclusions were developed in this framework by Ali *et al.* [3]. Abodayeh *et al.* [2] used Branciari type metric spaces to study hybrid contractive mappings. Pitea [19] employed the framework of dualistic partial metric spaces to study nonself mappings.

Later on, Kamran *et al.* [14] replaced the positive constant from the b-metric with a positive valued function, and so making a further generalization of b-metric spaces, which they called extended b-metric space. Extended comparison functions were used by Samreen *et al.* [21] to design nonlinear contractive inequalities. The study of Alqahtani *et al.* [4] points out the existence of common fixed points of certain mappings on extended b-metric spaces. In [1], Abdou and Alasmari introduced the concept of α-ψ-contractive mappings that led to fixed point theory results on extended b-metric spaces that helped in proving the existence of a solution to Fredholm integral equations. Ricinschi [20] proved existence results for fixed points of Istrăţescu type in the same setting.

The aim of this article is to state some conditions for the existence of ϕ-fixed points on extended b-metric spaces, using a class of operators that respect specific contractive inequalities and other properties including a monotone behaviour or continuity. These fixed points are a part of the set of zeros for a lower semi continuous function. The use of the extended b-metrics in this work is possible based on a generalization of the idea implemented by Lucács and Kajántó [18], which states a suitable method allowing us to pass from classic metric spaces to b-metric spaces with regard to implicit type of contractions. In our case, this transition is enabled by a condition implying a sequence that eventually converges to the fixed point.

2. Preliminaries

Definition 2.1 ([14])**.** Let X be a non-empty set and $\theta\colon X \times X \to [1,\infty)$ be a function. We say that $d\colon X \times X \to [0,\infty)$ is an extended b-metric if the following conditions are satisfied, for every x, y, $z \in X$:

(1) $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$;
(3) $d(x,y) \le \theta(x,y)\,[d(x,z) + d(z,y)]$.

In this case, (X,d) is called an extended b-metric space.

If the function θ is replaced by a constant $s \ge 1$, we obtain the definition of b-metric spaces. We underline that, in general, not all extended b-metric spaces are b-metric spaces, fact proved by the following example.

Example 2.1 ([4])**.** Considering $X = [0,1]$, we define a mapping

$$\theta\colon X \times X \to [1,\infty), \qquad \theta(x,y) = \frac{x+y+1}{x+y}.$$

Let

$$d\colon X \times X \to [0,\infty), \qquad d(x,y) = \begin{cases} \dfrac{1}{xy}, \text{ for } x,\ y \in (0,1],\ x \ne y; \\[2ex] 0, \text{ for } x,\ y \in [0,1],\ x = y; \\[2ex] \dfrac{1}{x}, \text{ for } x, \in (0,1],\ y = 0; \\[2ex] \dfrac{1}{y}, \text{ for } x = 0,\ y \in (0,1]. \end{cases}$$

We observe that the pair (X,d) is an extended b-metric space but is not a b-metric space. To prove that the mapping d is not a b-metric, presume that there exists a constant $s \ge 1$, so that $d(x,y) \le s(d(x,z) + d(z,y))$. Using the triangle inequality for the case of different values x, y, $z \in (0,1]$, from $d(x,y) \le s[d(x,z) + d(z,y)]$, we get $s \ge \dfrac{z}{x+y}$. Obviously, for $z = 1$, and

$x \neq y \to 0$, the inequality is false. Hence, the extended b-metric d is not a b-metric.

Definition 2.2 ([14])**.** Let (X, d) be an extended b-metric space. A sequence $\{x_n\} \subseteq X$ is convergent to $x \in X$ if for any $\varepsilon > 0$, there is $N = N(\varepsilon) \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$, for all $n \geq N$.

Definition 2.3 ([14])**.** Let (X, d) be an extended b-metric space. $\{x_n\}$ is a Cauchy sequence if for any $\varepsilon > 0$, there is $N = N(\varepsilon) \in \mathbb{N}$ such that $d(x_n, x_{n+p}) < \varepsilon$, for all $n \geq N$, and $p \geq 1$.

An extended b-metric space (X, d) is complete if any Cauchy sequence of X is convergent.

Apart from the classic metric spaces, which are continuous functions, extended b-metrics are not continuous (in this respect, we mention that b-metrics are not continuous, please see [14]). However, extended b-metrics enjoy the property of uniqueness of the limit of convergent sequences, please see [21].

3. **Implicit contractive conditions**

A type of operators which have good enough properties for the study of ϕ-fixed points is defined with the help of the next class of functions.

Definition 3.1. Denote by $\mathfrak{L}$ the class of functions $L\colon \mathbb{R}^2_+ \setminus \{(0,0)\} \to \mathbb{R}$ which fulfill the below hypotheses:

(M_1) for each sequences $\{a_n\}$, $\{b_n\}$, with $a_n \geq 0$, $b_n \geq 0$, $n \in \mathbb{N}$, we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$ if and only if $\lim_{n\to\infty} L(a_n, b_n) = -\infty$;

(M_2) for each sequences $\{a_n\}$, $\{b_n\}$, with $a_n \geq 0$, $b_n \geq 0$, $n \in \mathbb{N}$, so that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$, there exists $h \in (0,1)$ so that $\lim_{n\to\infty} a_n^h L(a_n, b_n) = 0$.

Example 3.1. Consider $L\colon \mathbb{R}^2_+ \setminus \{(0,0)\} \to \mathbb{R}$ with $L(x, y) = \ln(x + y)$. For every $h \in (0,1)$ the mapping L respects the axioms enunciated in the previous definition.

Definition 3.2 (Property S)**.** Let $L \in \mathfrak{L}$, and $k > 0$. The mapping L fulfills property S if for any sequence $\{x_n\} \subset (0, \infty)$ such that there exists $k > 0$ so that

$$k + L\left(\theta(x_n, x_{n+1}) d(x_n, x_{n+1}), \phi(x_n)\right) \leq L\left(d(x_{n-1}, x_n), \phi(x_{n-1})\right),$$

for all $n \geq 1$, it follows that

$$k + L\left(\prod_{i=1}^{n} \theta(x_i, x_{i+1}) d(x_n, x_{n+1}), \phi(x_n)\right) \leq L\left(\prod_{i=1}^{n-1} \theta(x_i, x_{i+1}) d(x_{n-1}, x_n), \phi(x_{n-1})\right), \text{ for all } n \geq 1.$$

Theorem 3.1. *Let (X, d) be a complete extended b-metric space, $\phi\colon X \to [0, \infty]$ be lower semi continuous, and $T\colon X \to X$ a mapping such that there exist $L \in \mathfrak{L}$ and $k > 0$, so that, for every x, $y \in X$, we have*

$$k + L\left(\theta(Tx, Ty) d(Tx, Ty), \phi(Tx)\right) \leq L\left(d(x, y), \phi(x)\right), \tag{1}$$

whenever $\min\{\theta(Tx, Ty)d(Tx, Ty) + \phi(Tx), d(x, y) + \phi(x)\} > 0$.

Suppose that for any sequence $x_{n+1} = Tx_n$, $n \in \mathbb{N}$, there is a constant $M > 0$ that satisfies the inequality $\dfrac{1}{\theta_m} \displaystyle\prod_{i=n}^{m} \theta(x_i, x_{m+1}) \leq M$, when m, $n \in \mathbb{N}$, $m \geq n$, and $\theta_m = \prod_{i=1}^{m} \theta(x_i, x_{i+1})$.

Then T has a ϕ-fixed point.

Proof. Without loss of generality, assume that $x_{n+1} \neq x_n$. Replacing $x = x_{n-1}$ and $y = x_n$ in the contractive condition (1), we get

$$k + L(\theta(x_n, x_{n+1}) d(x_n, x_{n+1}), \phi(x_n)) \leq L(d(x_{n-1}, x_n), \phi(x_n)), \quad n \geq 1.$$

By using the property S in the previous relation, we get, for all $n \geq 1$, that

$$k + L\left(\theta_n d(x_n, x_{n+1}), \phi(x_n)\right) \leq L\left(\theta_{n-1} d(x_{n-1}, x_n), \phi(x_{n-1})\right),$$

which will lead us to

$$L\left(\theta_n d(x_n, x_{n+1}), \phi(x_n)\right) \leq L(d(x_0, x_1), \phi(x_0)) - nk. \tag{2}$$

From this point up to the end of the proof, we will use the notations $d_n = d(x_n, x_{n+1})$, $\phi(x_n) = \phi_n$, $L_n = L(\theta_n d_n, \phi_n)$, $n \in \mathbb{N}$.

Taking the limit in inequality (2), we get $\lim_{n\to\infty} L_n = -\infty$ and using Definition 3.1 for the class of functions L, it follows that $\lim_{n\to\infty} \theta_n d_n = 0$ and $\lim_{n\to\infty} \phi_n = 0$.

From hypothesis (M_1), we get that exists a constant $h \in (0, 1)$ such that $\lim_{n\to\infty} \theta_n^h d_n^h L_n = 0$. Multiplying relation (2) with $\theta_n^h d_n^h$, we have

$$\theta_n^h d_n^h L_n - \theta_n^h d_n^h L_0 \leq -\theta_n^h d_n^h nk \leq 0,$$

which compels $\lim_{n\to\infty} \theta_n^h d_n^h n = 0$.

The last equality ensures the existence of $n_1 > 1$, such that $\theta_n^h d_n^h n \leq 1$, for all $n \geq n_1$. It follows that

$$d_n \leq \frac{1}{\theta_n n^{1/n}}, \quad \text{for all } n \geq 1.$$

This inequality involving d_n is going to help us prove that $\{x_n\}$ is a Cauchy sequence. Suppose $m, n \in \mathbb{N}$, $m > n$; applying the generalized triangle inequality, we have the following relations

$$\begin{aligned}
d(x_n, x_m) &\leq \theta(x_n, x_m)\left[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)\right] \\
&= \theta(x_n, x_m) d(x_n, x_{n+1}) + \theta(x_n, x_m)\theta(x_{n+1,x_m}) \\
&\quad [d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\
&\leq \theta(x_n, x_m) d(x_n, x_{n+1}) + \theta(x_n, x_m)\theta(x_{n+1}, x_m) d(x_{n+1}, x_{n+2}) + \dots \\
&\quad \dots + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\dots\theta(x_{m-1}, x_m) d(x_{m-1}, x_m) \\
&\leq d(x_n, x_{n+1}) \prod_{i=n}^{n} \theta(x_i, x_m) + d(x_{n+1}, x_{n+2}) \prod_{i=n}^{n+1} \theta(x_i, x_m) + \dots \\
&\quad \dots + d(x_{m-1}, x_m) \prod_{i=n}^{m-1} \theta(x_i, x_m) \\
&\leq \frac{1}{\theta_n n^{1/h}} \prod_{i=n}^{n} \theta(x_i, x_m) + \frac{1}{\theta_{n+1}(n+1)^{1/h}} \prod_{i=n}^{n+1} \theta(x_i, x_m) + \dots \\
&\quad \dots + \frac{1}{\theta_{m-1}(m-1)^{1/h}} \prod_{i=n}^{m-1} \theta(x_i, x_m) \\
&\leq \sum_{j=n}^{m-1} \frac{1}{\theta_j j^{1/h}} \prod_{i=n}^{j} \theta(x_i, x_m) \\
&\leq M \sum_{j=n}^{\infty} \frac{1}{j^{1/h}}.
\end{aligned}$$

We have shown that $\lim_{m,n\to\infty} d(x_n, x_m) = 0$, which proves the convergence of the sequence $\{x_n\}$ to $x \in X$.

Now, we aim to prove that x is the searched ϕ-fixed point. Knowing that the function ϕ is lower semi continuous, we have that $\lim_{n\to\infty} \phi(x_n) = 0$ and because $x_n \to x$, it follows that $\phi(x) = 0$. Denote $x^* = Tx$, and presume that $x^* \neq x$. Without loss of generality, we may assume that $x_n \neq x$, and $x_n \neq x^*$, $n \in \mathbb{N}$. Using the contractive condition, we have

$$L(\theta(x_{n+1}, x^*) d(x_{n+1}, x^*), \phi(x_{n+1})) \leq L(d(x_n, x), \phi(x_n)).$$

Because $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} \phi(x_n) = 0$, by applying the limit when $n \to \infty$ in the last inequality, we get

$$\lim_{n\to\infty} L(\theta(x_{n+1}, x^*)d(x_{n+1}, x^*), \phi(x_{n+1})) = -\infty,$$

so $\lim_{n\to\infty} \theta(x_{n+1}, x^*)d(x_{n+1}, x^*) = 0$. Because of the uniqueness of a limit in extended b-metric spaces, the limit is unique, we conclude that $x^* = x$ and x is a ϕ-fixed point of T. □

In order to state the last theorem on the existence of a ϕ-fixed point, we need a modified version of the class of function $\mathfrak{L}$ from Definition 3.1, in the sense that the constant h from axiom (M_2) will be smaller that 1/2.

Theorem 3.2. *Let (X, d) be a complete extended b-metric space, $\phi\colon X \to [0, \infty)$ be lower semi continuous, and $T\colon X \to X$ a mapping such that there exist $L \in \mathfrak{L}$, and $k > 0$, so that for every x, $y \in X$ we have*

$$k + L(\theta(Tx, Ty)d(Tx, Ty), \phi(Tx)) \le L(d(x, y), \phi(x)), \tag{3}$$

whenever $\min\{\theta(Tx, Ty)d(Tx, Ty) + \phi(Tx), d(x, y) + \phi(x)\} > 0$.

Furthermore, presume that for any sequence $\{x_{n+1} = Tx_n\} \subset X$, $n \in \mathbb{N}$, such that

$$\frac{1}{\theta_m} \prod_{i=n}^{m} \theta(x_i, x_{m+1}) \le n,$$

for any m, $n \in \mathbb{N}$, $m \ge n$, where $\theta_m = \prod_{i=1}^{m} \theta(x_i, x_{i+1})$.

Then T has a ϕ-fixed point.

Proof. Similar to the previous proof, we use the contractive condition (3) and the property S for $n = 2$ to obtain

$$L(\theta_2 d(x_2, x_3), \phi(x_2)) \le L(d(x_0, x_1), \phi(x_0)) - 2k.$$

Applying this process to $n \ge 1$, it follows that

$$L(\theta_n d(x_n, x_{n+1}), \phi(x_n)) \le L(d(x_0, x_1), \phi(x_0)) - nk. \tag{4}$$

The next step is introducing the following notations $d_n = d(x_n, x_{n+1})$, $\phi_n = \phi(x_n)$, $L_n = L(\theta_n d_n, \phi_n)$. With this notations in mind, relation (4) becomes

$$L_n \le L_0 - nk,$$

for $n \ge 1$ and by taking the limit in this inequality, we get $\lim_{n\to\infty} L_n = -\infty$. From axiom (M_1) it follows that $\lim_{n\to\infty} \theta_n d_n = 0$ and $\lim_{n\to\infty} \phi_n = 0$ and taking advantage of the last property from Definition 3.1 follows that there is $h < 1/2$ so that $\lim_{n\to\infty} \theta_n^h d_n^h L_n = 0$.

We can return to inequality (4), which imposes $\lim_{n\to\infty}\theta_n^h d_n^h n = 0$. Using this last relation, let $n_1 > 1$ so that $\theta_n^h d_n^h n \le 1$, when $n \ge n_1$. We get that

$$d_n \le \frac{1}{\theta n^{1/h}}, \quad \text{for all } n \in \mathbb{N}.$$

The next step is proving that $\{x_n\}$ converges to the searched ϕ-fixed point, which means that the sequence is a Cauchy one. Suppose that $m > n$, m, $n \in \mathbb{N}$ and using the last relation together with the triangle inequality, we have

$$\begin{aligned}
d(x_n, x_m) &\le \theta(x_n, x_m)\left[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)\right] \\
&= \theta(x_n, x_m) d(x_n, x_{n+1}) + \theta(x_n, x_m)\theta(x_{n+1,x_m}) \\
&\quad \left[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)\right] \\
&\le \theta(x_n, x_m) d(x_n, x_{n+1}) + \theta(x_n, x_m)\theta(x_{n+1}, x_m) d(x_{n+1}, x_{n+2}) + \dots \\
&\quad \dots + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\dots\theta(x_{m-1}, x_m) d(x_{m-1}, x_m) \\
&\le d(x_n, x_{n+1}) \prod_{i=n}^{n} \theta(x_i, x_m) + d(x_{n+1}, x_{n+2}) \prod_{i=n}^{n+1} \theta(x_i, x_m) + \dots \\
&\quad \dots + d(x_{m-1}, x_m) \prod_{i=n}^{m-1} \theta(x_i, x_m) \\
&\le \frac{1}{\theta_n n^{1/h}} \prod_{i=n}^{n} \theta(x_i, x_m) + \frac{1}{\theta_{n+1}(n+1)^{1/h}} \prod_{i=n}^{n+1} \theta(x_i, x_m) + \dots \\
&\quad \dots + \frac{1}{\theta_{m-1}(m-1)^{1/h}} \prod_{i=n}^{m-1} \theta(x_i, x_m) \\
&\le \sum_{j=n}^{m-1} \frac{1}{\theta_j j^{1/h}} \prod_{i=n}^{j} \theta(x_i, x_m) \\
&\le \sum_{j=n}^{\infty} \frac{1}{j^{1/h-1}}.
\end{aligned}$$

Applying the limit we obtain $\lim_{n,m\to\infty} d(x_n, x_m) = 0$, and thus we proved that $\{x_n\}$ is a Cauchy sequence and $x_n \to x \in X$.

Now, it must be shown that $x \in X$ is a ϕ-fixed point of T. From hypothesis (ii) and $\lim_{n\to\infty}\phi(x_n) = 0$, we get $\phi(x) = 0$, which means that x is a part of the set of zeros of ϕ.

Considering that there is $x^* = Tx$ and $x^* \neq x$, assuming that $x_n \neq x$ and $x_n \neq x^*$, $n \in \mathbb{N}$, it follows that

$$k + L(\theta(x_{n+1}, x^*) d(x_{n+1}, x^*), \phi(x_{n+1}), \varphi(x^*)) \le L(d(x_n, x), \phi(x_n), \varphi(x)),$$

for all $n \in \mathbb{N}$.

Taking the limit when $n \to \infty$ in the last inequality and since $\lim_{n\to\infty} d(x_n, x) = 0$, $\lim_{n\to\infty} \phi(x_n) = 0$, $\varphi(x) = 0$, we obtain

$$\lim_{n\to\infty} L(\theta(x_{n+1}, x^*)d(x_{n+1}, x^*), \phi(x_{n+1}), \varphi(x^*)) = -\infty,$$

and $\lim_{n\to\infty} \theta(x_{n+1}, x^*)d(x_{n+1}, x^*) = 0$, so $x = x^*$. We proved that the point we initially found $x \in X$ is a ϕ-fixed point of the operator T. $\square$

4. **Conclusions**

In this work, we studied the existence of fixed points of mappings with adequate properties such as monotony and continuity and which, additionally, belong to a set of zeros of a lower semi continuous function. The operators in view here are bound to satisfy inequalities defined by means of auxiliary functions which fulfill adequate conditions. The setting chosen here is a general one, that of extended b-metric spaces.

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