A DOUBLE INERTIAL SELF-ADAPTIVE SUBGRADIENT EXTRAGRADIENT METHOD WITH CONJUGATE GRADIENT DIRECTION FOR VARIATIONAL INEQUALITIES

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In this paper, we propose a novel double inertial self-adaptive subgradient extragradient method with conjugate gradient direction for solving variational inequality problems involving quasi-monotone and Lipschitz continuous operators in real Hilbert spaces. The proposed algorithm incorporates the technique of double inertial steps combined with a conjugate gradient direction to accelerate the convergence speed. A remarkable feature of our method is the use of a self-adaptive step size strategy, which does not require prior knowledge of the Lipschitz constant of the cost operator. Under some suitable conditions, we establish the weak convergence theorem of the proposed algorithm. Finally, several numerical results are presented to illustrate the feasibility and superior efficiency of our method compared to other algorithms.

Keywords: variational inequality, subgradient extragradient method, double inertial steps, conjugate gradient direction

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1. Introduction

The variational inequality problem (VIP) is to find a point $x^* \in C$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C,$$

where H is a real Hilbert space, C is a nonempty closed convex subset of H, and $F : H \to H$ is an operator. The solution set of VIP is denoted by S.

The variational inequality problem provides a broad framework that encompasses various mathematical models, such as nonlinear programming, network equilibrium, and complementarity problems, which are essential in fields like engineering and economics (see [5, 7, 11, 13, 18, 20, 24, 27]). To address the VIP across its various applications, many solution methods have been proposed in the literature. One of the most prominent techniques is the extragradient method (EGM) by Korpelevich

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in [12], defined as follows:

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \lambda_n F(x_n)), \\ x_{n+1} = P_C(x_n - \lambda_n F(y_n)), \end{cases}$$

where $\lambda \in (0, \frac{1}{L})$ and F is a monotone operator. It is well-known that the sequence $\{x_n\}$ generated by the EGM converges weakly to a solution of the VIP. However, a significant drawback of this method is its computational cost. Specifically, both the operator F and the metric projection P_C must be evaluated twice at each iteration. Furthermore, the step size $\{\lambda_n\}$ relies heavily on the Lipschitz constant L of the underlying operator. This dependency makes the method computationally expensive and less practical, especially in scenarios where a closed-form expression for P_C is unavailable or when the Lipschitz constant is unknown a priori.

To mitigate the computational drawbacks associated with Algorithm (2), Censor et al. [3] proposed the subgradient extragradient method (SEGM). This approach modifies the EGM by substituting the second projection onto C with a projection onto a specific, constructible half-space. The algorithm is defined as follows: Given $x_0 \in C$, $L > 0$, and $\lambda_n \in (0, \frac{1}{L})$,

$$\begin{cases} y_n = P_C(x_n - \lambda_n Ax_n) \\ T_n = \{w \in H : \langle x_n - \lambda_n Ax_n - y_n, w - y_n \rangle \leq 0\} \\ x_{n+1} = P_{T_n}(x_n - \lambda_n Ay_n), \end{cases}$$

where weak convergence has been established under the assumption that the operator A is monotone and L-Lipschitz continuous. Following this, several enhanced variants of the SEGM have been introduced in the literature (see, for example, [4, 21, 23]).

Over the years, numerous enhanced variants of the SEGM have been developed for addressing variational inequalities, equilibrium problems, and various optimization models. In this line of research, inertial techniques have been incorporated to accelerate the convergence behavior of SEGM-type algorithms. Meanwhile, other studies have focused on designing schemes whose step sizes can be selected without requiring prior knowledge of Lipschitz constants and without relying on any line-search strategy. Yao et al. [26] introduced the following relaxed SEGM with double inertial steps

$$\begin{cases} w_n = x_n + \omega(x_n - x_{n-1}), \\ z_n = x_n + \gamma_n(x_n - x_{n-1}), \\ y_n = P_C(z_n - \lambda_n F z_n), \\ T_n = \{v \in H : \langle z_n - \lambda_n F z_n - y_n, v - y_n \rangle \leq 0\}, \\ x_{n+1} = (1 - \beta_n) z_n + \beta_n P_{T_n}(z_n - \lambda_n F y_n), \end{cases}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{ \frac{\mu\|z_n - y_n\|}{\|Fz_n - Fy_n\|}, \lambda_n \right\}, & \text{if } Fz_n \neq Fy_n, \\ \lambda_n, & \text{otherwise.} \end{cases}$$

Conjugate gradient-type methods have been extensively studied to solve variational inequality problems (see [1, 8–10, 17]). Arzuka et al. [2] introduced an extragradient technique with a conjugate gradient-type direction to solve monotone variational inequality problems in Hilbert spaces:

$$\begin{cases} \Theta_n = \dfrac{\Gamma_n}{max\{||d_n||, \psi\}}, \\ d_{n+1} = -F(x_n) + \Theta_n d_n, \\ y_n = P_C(x_n + \lambda_n d_{n+1}), \\ x_{n+1} = P_C(x_n - \lambda_n F(y_n)). \end{cases}$$

Motivated by [2, 22, 26], in this paper we relax these stringent assumptions and propose a novel double inertial self-adaptive subgradient extragradient method with conjugate gradient direction (DI-SEGM-CG) for solving quasi-monotone variational inequality problems in real Hilbert spaces. Under some suitable conditions, we establish the weak convergence theorem of the proposed algorithm. Our results improve and extend the corresponding results in [19, 26].

2. Preliminaries

In this section, we recall some definitions and lemmas.

Definition 2.1. *Let $F : H \to H$ be an operator. F is said to be*

(i) *L-Lipschitz continuous, if there exists a constant $L > 0$ such that*

$$\|Fx - Fy\| \leq L\|x - y\|, \forall x, y \in H.$$

(ii) *ϱ-strongly monotone, if there exists a constant $\varrho > 0$ such that*

$$\langle Fy - Fx, y - x\rangle \geq \varrho\|y - x\|^2, \forall x, y \in H.$$

(iii) *monotone, if*

$$\langle Fy - Fx, y - x\rangle \geq 0, \forall x, y \in H.$$

(iv) *η-strongly pseudo-monotone, if there exists a constant $\eta > 0$ such that*

$$\langle Fx, y - x\rangle \geq 0 \Rightarrow \langle Fy, y - x\rangle \geq \eta\|y - x\|^2, \forall x, y \in H.$$

(v) *pseudo-monotone, if*

$$\langle Fx, y - x\rangle \geq 0 \Rightarrow \langle Fy, y - x\rangle \geq 0, \forall x, y \in H.$$

(vi) *quasi-monotone, if*

$$\langle Fx, y - x\rangle > 0 \Rightarrow \langle Fy, y - x\rangle \geq 0, \forall x, y \in H.$$

Minty formulation of VIP (MVIP)is defined as: find $v \in C$ such that

$$\langle Fy, y - q\rangle \geq 0, \forall q \in C.$$

The solution set of MVIP is denoted by S_D. When F is quasi-monotone, we have S_D is a closed and convex subset of C. Furthermore, since C is convex and F is continuous, we have $S_D \subset S$.

Lemma 2.1 ([1]). *The mapping $P_C : H \to C$ which assigns to each point $u \in H$ the unique point $w \in C$ such that*

$$\|u - w\| \leq \|u - y\|, \forall y \in C$$

is called the metric projection of H onto C. Some properties of the mapping P_C include:

(i) $\|P_C(x) - P_C(y)\| \leq \|x - y\|, \quad \forall x, y \in H.$
(ii) $\langle x - P_C(x), y - P_C(x)\rangle \leq 0, \quad \forall y \in C$ *and* $x \in H.$

Lemma 2.2 ([25])**.** *The following statements hold in H*

(i) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \forall x, y \in H.$
(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \forall x, y \in H.$
(iii) $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \forall x, y \in H, \lambda \in \mathbb{R}.$

Lemma 2.3 ([16])**.** *If a sequence $(x_k)_{k\in\mathbb{N}} \subset H$ convergence weakly to x, then for any $y \neq x \in H$,*

$$\liminf_{n\to\infty} \|x_k - y\| > \liminf_{n\to\infty} \|x_k - x\|.$$

Lemma 2.4 ([15])**.** *Let $\{\phi_n\}$, $\{\delta_n\}$ and $\{\theta_n\}$ be sequences in $[0, +\infty)$ such that*

$$\phi_{n+1} \leq \phi_n + \theta_n(\phi_n - \phi_{n-1}) + \delta_n, \forall n \geq 1, \sum_{n=1}^{\infty} \delta_n < +\infty,$$

and exists a real number θ with $0 \leq \theta_n \leq \theta < 1$ for all $n \in N$. Then the following assertions hold

(i) $\sum_{n=1}^{\infty}[\phi_n - \phi_{n-1}]_+ < +\infty$ *where* $[t]_+ = \max\{t, 0\}$ *for any* $t \in R$*;*
(ii) *there exists* $\phi^* \in [0, +\infty)$ *such that* $\lim_{n\to\infty} \phi_n = \phi^*$*.*

Lemma 2.5 ([27])**.** *Let C be a nonempty subset of H and let $\{x_n\}$ be a sequence in H such that the following two conditions hold*

(i) *for each $x \in C$, the limit of sequence $\{\|x_n - x\|\}$ exists;*
(ii) *any weak cluster point of sequence $\{x_n\}$ is in C.*

Then there exists $x^* \in C$ such that $\{x_n\}$ converges weakly to x^*.

3. Proposed Algorithm

In this section, we state the assumptions and our algorithm.

Assumption 3.1.

(A1) The feasible set C is a nonempty, closed, and convex subset of H.
(A2) The operator $F : H \to H$ is quasi-monotone.
(A3) The operator $F : H \to H$ is L-Lipschitz continuous and satisfies the following condition:

$$\text{whenever } \{x_n\} \subset C \text{ and } x_n \rightharpoonup z, \text{ one has } \|Fz\| \leq \liminf_{n\to\infty} \|Fx_n\|.$$

(A4) $S_D \neq \emptyset$.

Algorithm 3.1. Choose the parameters $\mu \in (0, 1)$, $\lambda_1 > 0$, $p \in [0, 1]$, $\theta \in [0, 1]$ and $\psi > 0$. Let $u_0, u_1 \in H$ be given starting points and $\{\Gamma_n\}$, $\{\xi_n\}$ be two nonnegative real numbers sequences in $[0, 1]$ such that $\lim_{n\to\infty} \Gamma_n = 0$ and $\sum_{n=1}^{\infty} \xi_n < +\infty$. Set $d_0 = -F(u_0)$, $n := 1$. Determine

$$\beta_n = \frac{\Gamma_n}{\max(\|d_n\|, \psi)}, \tag{1}$$

and

$$d_{n+1} = -F(w_n) + \beta_n d_n. \tag{2}$$

Compute

$$\begin{cases} z_n = u_n + \theta(u_n - u_{n-1}), \\ w_n = u_n + p(u_n - u_{n-1}), \\ y_n = P_C(w_n + \lambda_n d_{n+1}), \\ s_n = P_{T_n}(w_n - \lambda_n F(y_n)), \end{cases} \tag{3}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\mu\frac{\|w_n-y_n\|^2+\|s_n-y_n\|^2}{2\langle F(w_n)-F(y_n),s_n-y_n\rangle}, \lambda_n + \xi_n\right\}, & \text{if } \langle F(w_n) - F(y_n), s_n - y_n\rangle > 0 \\ \lambda_n + \xi_n, & \text{otherwise,} \end{cases} \tag{4}$$

and T_n is given by

$$T_n := \{v \in H : \langle w_n + \lambda_n d_{n+1} - y_n, v - y_n\rangle \leq 0\}$$

If $w_n = y_n = u_n$, then stop. Otherwise,
Compute

$$u_{n+1} = (1-\tau)z_n + \tau s_n, \quad n \geq 1, \tag{5}$$

Set $n \leftarrow n + 1$, and go to (1).

Remark 3.1. *Observe that if $u_n = w_n = y_n$ which implies $u_n = P_C(u_n + \lambda_n d_{n+1})$, so we know $u_n \in S$.*

4. **Convergence Analysis**

Here we establish the weak convergence of the proposed algorithm 3.1.

Lemma 4.1 ([14])**.** *The sequence $\{\lambda_n\}$ generated by (4) is well-defined. Furthermore, $\lim_{n\to\infty}\lambda_n = \lambda$, where λ lies in the interval $[\min\{\frac{\mu}{L}, \lambda_1\}, \lambda_1 + \sum_{n=1}^{\infty}\xi_n]$.*

Lemma 4.2. *Suppose that assumption 3.1 holds and $0 < \tau < \frac{1}{1+\delta}$, $0 \leq \theta < \frac{\delta-\sqrt{2\delta}}{\delta}$ for some $\delta > 2$, Then, the sequence $\{u_n\}$ generated by algorithm 3.1 is bounded and $\lim_{n\to\infty}||u_n - x^*||$ exists, where $x^* \in S_D$.*

Proof. Let $x^* \in S_D$. Since $S_D \subseteq S$, x^* is a solution to the variational inequality. As $y_n \in C$, we have $\langle F(y_n), y_n - x^*\rangle \geq 0$, then

$$\langle F(y_n), y_n - x^* + s_n - s_n\rangle \geq 0,$$

and thus

$$\langle F(y_n), x^* - s_n\rangle \leq \langle F(y_n), y_n - s_n\rangle.$$

Using the property of the projection $s_n = P_{T_n}(w_n - \lambda_n F(y_n))$ and the definition of the half-space T_n which involves the conjugate gradient direction d_n, we have

$$\langle w_n + \lambda_n d_{n+1} - y_n, s_n - y_n\rangle \leq 0. \tag{6}$$

Substituting the update rule $d_{n+1} = -F(w_n) + \beta_n d_n$ into (6), we obtain

$$\langle w_n - y_n, s_n - y_n\rangle \leq \lambda_n\langle F(w_n), s_n - y_n\rangle - \lambda_n\langle\beta_n d_n, s_n - y_n\rangle.$$

From $y_n = P_c(w_n + \lambda_n d_n)$ and Lemma 2.1, we have

$$\langle w_n + \lambda_n d_n - y_n, y - y_n\rangle \leq 0, \forall y \in C,$$

so, $y \in T_n$, therefore $x^* \in C \in T_n$.

By Lemma 2.2, we have

$$
\begin{aligned}
||s_n - x^*||^2 &= ||P_{T_n}(w_n - \lambda_n F(y_n)) - P_{T_n} x^*||^2 \\
&\leq \langle s_n - x^*, w_n - \lambda_n F(y_n) - x^* \rangle \\
&= \frac{1}{2}||s_n - x^*||^2 + \frac{1}{2}||w_n - x^*||^2 - \frac{1}{2}||s_n - w_n||^2 - \langle s_n - x^*, \lambda_n F(y_n) \rangle,
\end{aligned}
$$

which implies that

$$
||s_n - x^*||^2 \leq ||w_n - x^*||^2 - ||s_n - w_n||^2 + 2\langle x^* - s_n, \lambda_n F(y_n) \rangle. \tag{7}
$$

Based on $x^* \in S_D$ and (7), we get

$$
\begin{aligned}
||s_n - x^*||^2 &\leq ||w_n - x^*||^2 - ||s_n - w_n||^2 + 2\lambda_n \langle F(y_n), y_n - s_n \rangle \\
&= ||w_n - x^*||^2 - ||s_n - w_n||^2 - ||w_n - y_n||^2 \\
&\quad + 2\langle w_n - \lambda_n F(y_n) - y_n, s_n - y_n \rangle.
\end{aligned} \tag{8}
$$

Since $y_n = P_C(w_n + \lambda_n d_n)$ and $s_n \in T_n$, we have

$$
\begin{aligned}
2\langle w_n - \lambda_n F(y_n) - y_n, s_n - y_n \rangle &= 2\langle w_n + \lambda_n d_{n+1} - y_n, s_n - y_n \rangle \\
&\quad + 2\lambda_n \langle d_{n+1} + F(w_n), y_n - s_n \rangle + 2\langle \lambda_n F(w_n) - \lambda_n F(y_n), s_n - y_n \rangle \\
&\leq 2\lambda_n \beta_n \langle d_n, y_n - s_n \rangle + 2\lambda_n \langle F(w_n) - F(y_n), s_n - y_n \rangle \\
&\leq 2\lambda_n \langle F(w_n) - F(y_n), s_n - y_n \rangle + 2\lambda_n \Gamma_n ||y_n - s_n||
\end{aligned} \tag{9}
$$

By (8) and (9), we get

$$
\begin{aligned}
||s_n - x^*||^2 &\leq ||w_n - x^*||^2 - ||s_n - w_n||^2 - ||w_n - y_n||^2 \\
&\quad + 2\lambda_n \langle F(w_n) - F(y_n) - y_n, s_n - y_n \rangle + 2\lambda_n \Gamma_n ||y_n - s_n|| \\
&\leq ||w_n - x^*||^2 - ||s_n - w_n||^2 - ||w_n - y_n||^2 \\
&\quad + \frac{\lambda_n \mu}{\lambda_{n+1}}(||w_n - y_n||^2 + ||s_n - y_n||^2) + \lambda_n \Gamma_n ||y_n - s_n||^2 + \lambda_n \Gamma_n \\
&\leq ||w_n - x^*||^2 - (1 - (\frac{\lambda_n \mu}{\lambda_{n+1}} + \lambda_n \Gamma_n))(||w_n - y_n||^2 + ||s_n - y_n||^2) + \lambda_n \Gamma_n.
\end{aligned}
$$

Since $\lim_{n\to\infty} \lambda_n = \lambda$ and $\lim_{n\to\infty} \Gamma_n = 0$, one has $\lim_{n\to\infty}(1 - (\frac{\lambda_n \mu}{\lambda_{n+1}} + \lambda_n \Gamma_n))$ $= 1 - \mu > 0$. Then there exists a natural number $N \geq 1$ such that

$$
||s_n - x^*||^2 \leq ||w_n - x^*||^2 + \lambda_n \Gamma_n.
$$

Thanks to (5), we have

$$
\begin{aligned}
||u_{n+1} - x^*||^2 &= ||(1-\tau)(z_n - x^*) + \tau(s_n - x^*)||^2 \\
&= (1-\tau)||z_n - x^*||^2 + \tau||s_n - x^*||^2 - \tau(1-\tau)||z_n - s_n||^2 \\
&\leq (1-\tau)||z_n - x^*||^2 + \tau\left(||w_n - x^*||^2 + \lambda_n \Gamma_n\right) - \tau(1-\tau)||z_n - s_n||^2,
\end{aligned} \tag{10}
$$

and

$$
||s_n - z_n||^2 \leq \frac{1}{\tau}||u_{n+1} - z_n||^2. \tag{11}
$$

Combining (10) with (11) to deduce

$$\begin{aligned}||u_{n+1}-x^*||^2 &\le (1-\tau)||z_n-x^*||^2+\tau||w_n-x^*||^2+\tau\lambda_n\Gamma_n-\frac{1-\tau}{\tau}||u_{n+1}-z_n||^2\\ &\le (1-\tau)||z_n-x^*||^2+\tau||w_n-x^*||^2-\frac{1-\tau}{\tau}||u_{n+1}-z_n||^2+\lambda_n\Gamma_n. \quad (12)\end{aligned}$$

From Lemma 2.2 and (3), we have

$$\begin{aligned}||z_n-x^*||^2 &= ||u_n+\theta(u_n-u_{n-1})-x^*||^2\\ &= ||(1+\theta)(u_n-x^*)-\theta(u_{n-1}-x^*)||^2\\ &= (1+\theta)||u_n-x^*||^2-\theta||u_{n-1}-x^*||^2+\theta(1+\theta)||u_n-u_{n-1}||^2, \quad (13)\end{aligned}$$

$$\begin{aligned}||w_n-x^*||^2 &= ||u_n+p(u_n-u_{n-1})-x^*||^2\\ &= ||(1+p)(u_n-x^*)-p(u_{n-1}-x^*)||^2\\ &= (1+p)||u_n-x^*||^2-p||u_{n-1}-x^*||^2+p(1+p)||u_n-u_{n-1}||^2, \quad (14)\end{aligned}$$

and

$$\begin{aligned}||u_{n+1}-z_n||^2 &= ||u_{n+1}-(u_n+\theta(u_n-u_{n-1}))||^2\\ &= ||(u_{n+1}-u_n)||^2+\theta^2||u_n-u_{n-1}||^2-2\theta\langle u_{n+1}-u_n,u_n-u_{n-1}\rangle\\ &\ge ||(u_{n+1}-u_n)||^2+\theta^2||u_n-u_{n-1}||^2-2||u_{n+1}-u_n||||u_n-u_{n-1}||\\ &\ge (1-\theta)||(u_{n+1}-u_n)||^2+(\theta^2-\theta)||u_n-u_{n+1}||^2. \quad (15)\end{aligned}$$

Substituting (13), (14) and (15) into (12), we have

$$\begin{aligned}||u_{n+1}-x^*||^2 &\le (1-\tau)[(1+\theta)||u_n-x^*||^2-\theta||u_{n-1}-x^*||^2+\theta(1+\theta)||u_n-u_{n-1}||^2]\\ &\quad +\tau[(1+p)||u_n-x^*||^2-p||u_{n-1}-x^*||^2+p(1+p)||u_n-u_{n-1}||^2]\\ &\quad -\frac{1-\tau}{\tau}[(1-\theta)||u_{n+1}-u_n||^2+(\theta^2-\theta)||u_n-u_{n-1}||^2]+\lambda_n\Gamma_n\\ &= [1+\tau p+\theta(1-\tau)]||u_n-x^*||^2-[(1-\tau)\theta+\tau p]||u_{n-1}-x^*||^2\\ &\quad +[(1-\tau)\theta(1+\theta)+\tau p(1+p)-\frac{(1-\tau)(\theta^2-\theta)}{\tau}]||u_n-u_{n-1}||^2\\ &\quad -\frac{(1-\tau)(1-\theta)}{\tau}||u_{n+1}-u_n||^2+\lambda_n\Gamma_n\\ &\le ||u_n-x^*||^2+[(1-\tau)\theta+\tau p](||u_n-x^*||^2-||u_{n-1}-x^*||^2)\\ &\quad +[(1-\tau)\theta(1+\theta)+\tau p(1+p)-\frac{(1-\tau)(\theta^2-\theta)}{\tau}]||u_n-u_{n-1}||^2+\lambda_n\Gamma_n. \quad (16)\end{aligned}$$

Let

$$\begin{aligned}\Omega_n &= ||u_n-x^*||^2-(\tau p+\theta(1-\tau))||u_{n-1}-x^*||^2\\ &\quad +[(1-\tau)\theta(1+\theta)+\tau p(1+p)-\frac{(1-\tau)(\theta^2-\theta)}{\tau}]||u_n-u_{n-1}||^2. \quad (17)\end{aligned}$$

In view of (16) and (17), we obtain

$$\Omega_{n+1}-\Omega_n \le -[\frac{1-\tau}{\tau}(\theta-1)^2-(1-\tau)\theta(1+\theta)-\tau p(1+p)]||u_{n+1}-u_n||^2+\lambda_n\Gamma_n. \quad (18)$$

Since $\tau < \frac{1}{1+\delta}$, $\frac{1-\tau}{\tau} > \delta$. Then,

$$\begin{aligned}\frac{1-\tau}{\tau}(\theta-1)^2 - (1-\tau)\theta(1+\theta) - \tau p(1+p) &> \delta(\theta-1)^2 - 2(1-\tau) - 2\tau \\ &= \delta\theta^2 - 2\delta\theta + \delta - 2. \qquad (19)\end{aligned}$$

From $0 \le \theta < \frac{\delta - \sqrt{2\delta}}{\delta}$, we can get

$$\delta\theta^2 - 2\delta\theta + \delta - 2 > 0.$$

By virtue of (18) and (19), we have

$$\Omega_{n+1} - \Omega_n < -(\delta\theta^2 - 2\delta\theta + \delta - 2)||u_{n+1} - u_n||^2 + \lambda_n\Gamma_n$$

Noting that $\lim_{n\to\infty} \Gamma_n = 0$, we deduce

$$\Omega_{n+1} - \Omega_n \le -(\delta\theta^2 - 2\delta\theta + \delta - 2)||u_{n+1} - u_n||^2 \le 0.$$

Hence, Ω_n is non-increasing. Therefore,

$$\begin{aligned}\Omega_n &= ||u_n - x^*||^2 - (\tau p + \theta(1-\tau))||u_{n-1} - x^*||^2 \\ &+ [(1-\tau)\theta(1+\theta) + \tau p(1+p) - \frac{(1-\tau)(\theta^2-\theta)}{\tau}]||u_n - u_{n-1}||^2 \\ &\ge ||u_n - x^*||^2 - (\tau p + \theta(1-\tau))||u_{n-1} - x^*||^2 \\ &= ||u_n - x^*||^2 - \epsilon||u_{n-1} - x^*||^2. \qquad (20)\end{aligned}$$

Write $\epsilon = \tau p + \theta(1-\tau)$ and we have $\epsilon \le max[\theta, p] < 1$. From (20), we have

$$\begin{aligned}||u_n - x^*||^2 &\le \epsilon||u_{n-1} - x^*||^2 + \Omega_n \\ &\le \epsilon||u_{n-1} - x^*||^2 + \Omega_N \\ &\le \epsilon^{n-N}||u_N - x^*||^2 + \frac{\Omega_N}{1-\epsilon}. \qquad (21)\end{aligned}$$

Take into account of (20), we also have

$$\Omega_{n+1} \ge ||u_{n+1} - x^*||^2 - \epsilon||u_n - x^*||^2 \ge -\epsilon||u_n - x^*||^2. \qquad (22)$$

By (21) and (22), we derive

$$-\Omega_{n+1} \le \epsilon||u_n - x^*||^2 \le \epsilon^{n-N+1}||u_N - x^*||^2 + \frac{\epsilon\Omega_N}{1-\epsilon}.$$

Then,

$$\begin{aligned}(\delta\theta^2 - 2\delta\theta + \delta - 2)\sum_{n=N}^{k}||u_{n+1} - u_n||^2 &\le \Omega_N - \Omega_{k+1} + \sum_{n=N}^{k}\lambda_n\Gamma_n \\ &\le \epsilon^{k-N+1}||u_N - x^*||^2 + \frac{\Omega_N}{1-\epsilon} + \sum_{n=N}^{k}\lambda_n\Gamma_n \\ &\le ||u_N - x^*||^2 + \frac{\Omega_N}{1-\epsilon} + \sum_{n=N}^{k}\lambda_n\Gamma_n, \forall k > N,\end{aligned}$$

which implies $\sum_{n=1}^{\infty}||u_{n+1} - u_n||^2 \le +\infty$, and it follows that

$$\lim_{n\to\infty}||u_{n+1} - u_n|| = 0.$$

From (16), we have

$$\begin{aligned}||u_{n+1}-x^*||^2 &\leq ||u_n-x^*||^2+[(1-\tau)\theta+\tau p](||u_n-x^*||^2-||u_{n-1}-x^*||^2)\\ &+[(1-\tau)\theta(1+\theta)+\frac{2}{1+\delta}+\frac{1-\tau}{4\tau}]||u_n-u_{n-1}||^2+\lambda_n\Gamma_n, \forall n\geq N \qquad (23)\end{aligned}$$

Applying Lemma 2.4 to (23), we can get $\lim_{n\to\infty}||u_n-x^*||$ exists. Therefore $\{u_n\}$ is bounded. □

Lemma 4.3. *Let $\{u_n\}$ be generated by Algorithm 3.1. Suppose that the assumption 3.1 holds. If q^* is one of the weak cluster points of $\{u_n\}$, then $q^*\in S_D$ or $F(q^*)=0$.*

Proof. By Lemma 4.2, we have $\{u_n\}$ is bounded, so let q^* be a weak cluster point of $\{u_n\}$, then we can choose a subsequence of $\{u_n\}$ denoted by $\{u_{n_k}\}$ such that $u_{n_k}\to q^*\in H$.

In the following proof, we first consider two distinct cases.

Case A : If $\lim_{k\to\infty}\sup||F(u_k)||=0$, then $\lim_{k\to\infty}||F(u_k)||=\lim_{k\to\infty}\inf||F(u_k)||=0$, so from $(A2)$, we can get that $F(q^*)=0$.

Case B: If $\lim_{k\to\infty}\sup||F(u_k)||>0$, then we can choose a subsequence of $F(u_{n_k})$ still denoted by $F(u_{n_k})$ such that $\lim_{k\to\infty}||F(u_{n_k})||=M_1>0$.

From Lemma 4.2, we have $\lim_{n\to\infty}||u_{n+1}-u_n||=0$. Then, we deduce

$$\begin{aligned}||u_{n+1}-z_n|| &= ||u_{n+1}-u_n-\theta(u_n-u_{n-1})||\\ &\leq ||u_{n+1}-u_n||+\theta||u_n-u_{n-1}||\to 0, n\to\infty,\\ ||u_{n+1}-w_n|| &= ||u_{n+1}-u_n-p(u_n-u_{n-1})||\\ &\leq ||u_{n+1}-u_n||+p||u_n-u_{n-1}||\to 0, n\to\infty,\\ ||z_n-s_n|| &= \frac{1}{\tau}||u_{n+1}-z_n||\to 0, n\to\infty,\\ ||w_n-z_n|| &= ||(w_n-u_{n+1})+(u_{n+1}-z_n)||\\ &\leq ||w_n-u_{n+1}||+||u_{n+1}-z_n||\to 0, n\to\infty,\\ ||w_n-s_n|| &\leq ||w_n-z_n||+||z_n-s_n||\to 0, n\to\infty.\end{aligned}$$

Since $\{u_n\}$ is bounded and $||u_{n+1}-w_n||\to 0$, $||w_n-s_n||\to 0$, we can deduce that $\{||w_n-x^*||\}$ and $\{||s_n-x^*||\}$ are also bounded. Note that

$$\begin{aligned}&(1-(\frac{\mu\lambda_n}{\lambda_{n+1}}+\lambda_n\Gamma_n))(||w_n-y_n||^2+||s_n-y_n||^2)\\ &\leq ||w_n-x^*||^2-||s_n-x^*||^2+\lambda_n\Gamma_n\\ &= (||w_n-x^*||+||s_n-x^*||)||w_n-x^*||-||s_n-x^*||)+\lambda_n\Gamma_n\\ &\leq M||w_n-s_n||+\lambda_n\Gamma_n\to 0, n\to\infty.\end{aligned}$$

Since $\lim_{n\to\infty}\lambda_n=\lambda$, $\lim_{n\to\infty}\Gamma_n=0$, we have

$$\lim_{n\to\infty}||w_n-y_n||=\lim_{n\to\infty}||s_n-y_n||=0.$$

From $||u_{n+1}-w_n||\to 0$, $||w_n-y_n||\to 0$, $||u_{n+1}-u_n||\to 0$, we have $||u_n-y_n||\to 0, n\to\infty$, then we can choose a subsequence of $\{y_n\}$ and denoted by $\{y_{n_k}\}$ such that $y_{n_k}\to q^*\in H$. By the definition of y_n, $y_n=P_C(w_n+\lambda_n d_{n+1})$ and Lemma 2.1, we have

$$\langle w_n+\lambda_n d_{n+1}-y_n, y_n-y\rangle\geq 0, \forall y\in C.$$

Because $d_{n+1} = -F(w_n) + \beta_n d_{n-1}$, we obtain

$$\langle w_n - \lambda_n F(w_n) + \lambda_n \beta_n d_n - y_n, y_n - y\rangle \geq 0, \forall y \in C.$$

It follows that

$$\langle \frac{w_n - y_n}{\lambda_n} - F(w_n) + \beta_n d_n, y_n - y\rangle \geq 0, \forall y \in C.$$

Hence,

$$\begin{aligned}\langle \frac{w_n - y_n}{\lambda_n}, y_n - y\rangle + \beta_n\langle d_n, y_n - y\rangle + \langle F(y_n), y - y_n\rangle \\ + \langle F(w_n) - F(y_n), y - y_n\rangle \geq 0.\end{aligned}$$

This together with Lipschitz continuity of F implies that

$$\begin{aligned}\langle \frac{w_n - y_n}{\lambda_n}, y_n - y\rangle + \beta_n\langle d_n, y_n - y\rangle + \langle F(y_n), y - y_n\rangle \\ + L||w_n - y_n||||y - y_n|| \geq 0, \forall y \in C.\end{aligned}$$

So, we have

$$\begin{aligned}\langle \frac{w_{n_k} - y_{n_k}}{\lambda_{n_k}}, y_{n_k} - y\rangle + \beta_{n_k}\langle d_{n_k}, y_{n_k} - y\rangle + \langle F(y_{n_k}), y - y_{n_k}\rangle \\ + L||w_{n_k} - y_{n_k}||||y - y_{n_k}|| \geq 0, \forall y \in C.\end{aligned}$$

Since $\lim_{n\to\infty} ||w_n - y_n|| = 0$ and $\lim_{n\to\infty} \beta_n = 0$, we have

$$0 \leq \lim_{k\to\infty} \inf\langle F(y_{n_k}), y - y_{n_k}\rangle \leq \lim_{k\to\infty} \sup\langle F(y_{n_k}), y - y_{n_k}\rangle \leq \infty, \forall y \in C. \tag{24}$$

Base on (24), we distinguish two possibilities in Case B:
Case I: Suppose that $\lim_{k\to\infty} \sup\langle F(y_{n_k}), y - y_{n_k}\rangle > 0, \forall y \in C$, then we can choose a subsequence of $\{y_{n_k}\}$ denoted by $\{y_{n_{k_j}}\}$ such that

$$\lim_{j\to\infty} \langle F(y_{n_{k_j}}), y - y_{n_{k_j}}\rangle > 0,$$

so there exists $j_0 \geq 1$ such that

$$\langle F(y_{n_{k_j}}), y - y_{n_{k_j}}\rangle > 0, \forall j \geq j_0.$$

By the quasimonotonicity of F, we have

$$\langle Fy, y - y_{n_{k_j}}\rangle \geq 0, \forall j \geq j_0, y \in C.$$

Since $u_{n_k} \to q^* \in H$ and $||u_n - y_n|| \to 0, n \to \infty$, we have $y_{n_k} \to q^*$. Then,

$$\langle Fy, y - q^*\rangle \geq 0, y \in C,$$

therefore $q^* \in S_D$.
Case II: Suppose that $\lim_{k\to\infty} \sup\langle F(y_{n_k}), y - y_{n_k}\rangle = 0, \forall y \in C$. From (24), we have

$$\lim_{k\to\infty} \langle F(y_{n_k}), y - y_{n_k}\rangle = 0, \forall y \in C.$$

Next, we show the $q^* \in S_D$. We choose a sequence ϵ_k to be a positive decreasing numbers and $\lim_{k\to\infty} \epsilon_k = 0$, for each ϵ_k, let N_k be the smallest nonnegative integer such that

$$\langle F(y_{n_k}), y - y_{n_k}\rangle + \epsilon_k \geq 0, \forall k \geq N_k \tag{25}$$

Since $\{y_{n_k}\} \in C$, we can suppose that $F(y_{n_k}) \neq 0$ and define $V_{N_k} = \frac{F(y_{N_k})}{||F(y_{N_k})||^2}$. Thus, $\langle F(y_{N_k}), V_{N_k}\rangle = 1$.

Thanks to (25), we get

$$\langle F(y_{N_k}), y + \epsilon_k V_{N_k} - y_{N_k}\rangle \geq 0, \forall y \in C.$$

Using the quasi-monotonicity of F, we have

$$F(y + \epsilon_k V_{N_k}), y + \epsilon_k V_{N_k} - y_{N_k}\rangle \geq 0, \forall y \in C,$$

which implies that

$$\begin{aligned}\langle F(y), y + \epsilon_k V_{N_k} - y_{N_k}\rangle &\geq -||F(y) - F(y + \epsilon_k V_{N_k})|| \cdot ||y + \epsilon_k V_{N_k} - y_{N_k}|| \\ &\geq -L||\epsilon_k V_{N_k}|| \cdot ||y + \epsilon_k V_{N_k} - y_{N_k}|| \\ &= -\frac{L}{||F(y_{N_k})||}||\epsilon_k|| \cdot ||y + \epsilon_k V_{N_k} - y_{N_k}|| \\ &\geq \frac{-2L}{M_1}||\epsilon_k||M_2. \end{aligned} \tag{26}$$

Since $\|u_n - y_n|| \to 0$, $\lim_{k\to\infty} ||F(u_{n_k})|| = \lim_{k\to\infty} ||F(y_{n_k})|| = M_1 > 0$, we can find $k_j > 1$ such that $||F(y_{n_k})|| \geq \frac{M_1}{2}, \forall k \geq k_j$, $M_2 > 0$. According to (26), we can conclude that $\langle F(y), y - q^*\rangle \geq 0, \forall y \in C$. Hence $q^* \in S_D$. □

Theorem 4.1. *Suppose that assumption 3.1 holds and $F(y) \neq 0, \forall y \in C$. Then the sequence $\{u_n\}$ generated by algorithm 3.1 converges weakly to a point in S_D*

Proof. By the Lemma 4.2, we get $\{u_n\}$ is bounded. Suppose r is a weak cluster point of $\{u_n\}$, then we can choose a subsequence of $\{u_n\}$ denoted by $\{u_{n_k}\}$ such that $u_{n_k} \to r, k \to \infty$, from $||u_n - y_n|| \to 0, n \to \infty$, we can get $y_{n_k} \to r, k \to \infty$. Since C is closed, so $r \in C$. Since $F(y) \neq 0$, we have $F(r) \neq 0$. By Lemma 4.3, we have $r \in S_D$ and and every sequential weak cluster point of $\{u_n\}$ is in S_D. Therefore, applying Lemma 2.5, we get u_n converges weakly to a point in S_D. □

5. **Numerical Experiments**

In this section, we present numerical experiments to evaluate the performance of our proposed algorithm 3.1 by comparing it with several existing related algorithms. All simulations were implemented in MATLAB R2016a and executed on a desktop PC equipped with an Intel(R) Core(TM) i5-12500H CPU @ 2.50 GHz and 16.0 GB of RAM.

Throughout these examples, we compare the numerical performance of our proposed algorithm 3.1 against algorithm of Shehu et al. [19] and algorithm of Yao et al. [26]

Example 5.1. Following the numerical example provided by Dong et al. [6], we consider the operator $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined as:

$$F(x, y) = (2x + 2y + \sin(x), -2x + 2y + \sin(y)), \quad \forall (x, y) \in \mathbb{R}^2.$$

The authors Dong et al. showed that operator F is $\sqrt{26}$-Lipschitz continuous and strongly monotone. The feasible set is defined as a box constraint $C := \{x \in$

$\mathbb{R}^2 : -10 \leq x_i \leq 10, i = 1, 2\}$. The variational inequality problem possesses a unique solution at $x^* = (0, 0)$.

To conduct a fair and rigorous comparison, all algorithms are implemented in MATLAB on the same computing environment. The initial points x_0 and x_1 are randomly generated within the feasible set C and are kept identical for all algorithms in each trial to ensure consistent starting conditions. The maximum number of iterations is set to $n = 100$. The starting points x_0 and x_1 were randomly generated within the feasible set C and kept identical for all tested algorithms in each trial.

We take $\|x_n - x^*\|^2 < 10^{-4}$ as stopping criterion in Figures. The parameters for the compared algorithms are configured as follows based on the convergence conditions:

- **Our Alg. 1:** we take $\lambda_1 = 0.1, \delta = 2.1, \theta = 0.9\frac{\delta-\sqrt{2\delta}}{\delta}, p = 0.9, \mu = 0.8, \tau = \frac{0.9}{1+\delta}, \xi_n = \frac{0.5}{n^2}, \Gamma_n = \frac{1}{(n+1)^5}, \psi = 1$.
- **Shehu et al.:** We set $\lambda_1 = 0.1, \theta_n = 0.5$ and $\mu = 0.7, \delta = 2.1, \alpha_n = \frac{0.8}{1+\delta}$.
- **Yao et al.** : We set $\lambda_1 = 0.1, \theta_n = 0.8, \mu = 0.9, \epsilon = 2.1, \alpha_n = \frac{0.9}{1+\epsilon}$, $\delta = 0.9\frac{\epsilon-\sqrt{2\epsilon}}{\delta}$.

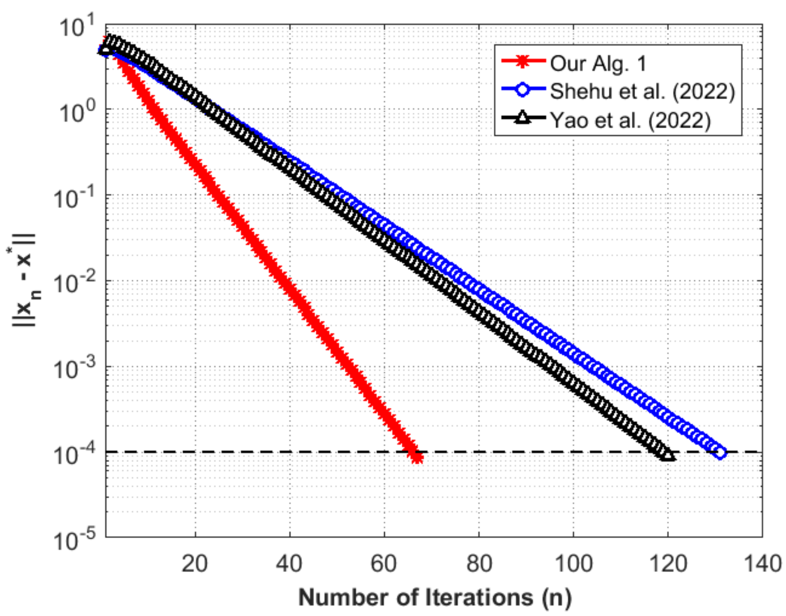

FIGURE 1. Comparison of number of iterations for Example 1.

TABLE 1. Number of iterations for Example 1

Algorithm	Alg. 1 in [19]	Alg. 1 in [26]	**our Alg. 1**
Number of iterations	131	120	**67**

The numerical results in Figure 1 and Table 1 demonstrate that the proposed algorithm outperforms the existing methods. Specifically, our algorithm achieves the target precision in just 67 iterations, significantly faster than Yao et al. (120) and Shehu et al. (131), confirming the efficiency of the conjugate gradient direction double inertial structure.

Example 5.2. We consider the nonlinear complementarity problem (NCP) examined by Liu and Yang [14], where the feasible set is given by $C = \mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x_i \geq 0,\ i = 1, \ldots, m\}$. The operator $F : \mathbb{R}^m \to \mathbb{R}^m$ is defined by $F(x) = (f_1(x), f_2(x), \ldots, f_m(x))^T$, with components:

$$f_i(x) = x_{i-1}^2 + x_i^2 + x_{i-1}x_i + x_i x_{i+1} - 2x_{i-1} + 4x_i + x_{i+1}, \quad i = 1, 2, \ldots, m,$$

where $x_0 = 0$ *and* $x_{m+1} = 0$.

In this experiment, we set the dimension $m = 10$. The optimization is performed with the stopping criterion $\|x_{n+1} - x_n\| < 10^{-6}$ or the maximum number of iterations $n = 300$. To evaluate the convergence behavior from a non-trivial starting position, the initial points x_0 and x_1 are chosen as vectors of all ones, i.e., $x_0 = x_1 = (1, 1, \ldots, 1)^T$.

The parameters for the compared algorithms are configured as follows based on the convergence conditions:

- **Our Alg. 1:** We choose $\mu = 0.4$, $\delta = 2.1$, $p = 1.0$, $\theta = 0.8\frac{\delta - \sqrt{2\delta}}{\delta}$, $\tau = \frac{0.9}{1+\delta}$, and $\xi_n = \frac{1}{n^2}, \psi = 1, \Gamma_n = \frac{1}{(n+1)^5}$.
- **Shehu et al.:** We set $\mu = 0.4$, $\delta = 2.1$, $\theta_n = 0.9\frac{\delta - \sqrt{2\delta}}{\delta}$, and $\alpha_n = \frac{0.9}{1+\delta}$.
- **Yao et al. :** We set $\mu = 0.4$, $\epsilon = 2.1$, $\delta = 0.9\frac{\epsilon - \sqrt{2\epsilon}}{\epsilon}$, $\theta_n = 1.0$, and $\alpha_n = \frac{0.9}{1+\epsilon}$.

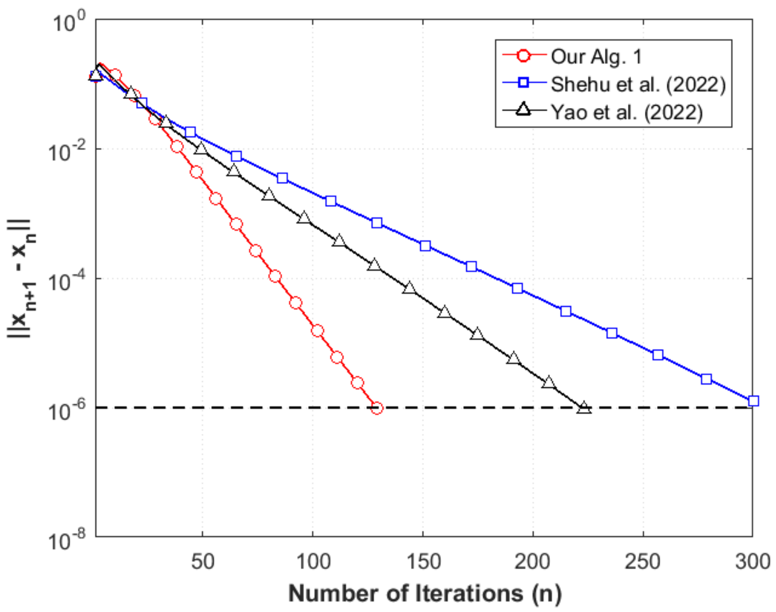

Figure 2. Comparison of number of iterations for Example 2.

TABLE 2. Number of iterations for Example 2

Algorithm	Alg. 1 in [19]	Alg. 1 in [26]	**Our Alg. 1**
Number of iterations	300	223	**129**

The Figure 2 and Table 2 reports the specific number of iterations required to reach the tolerance 10^{-6}. It is evident that the proposed Algorithm 1 (red line) significantly outperforms the methods of Shehu et al. and Yao et al., achieving the fastest convergence speed. Specifically, our method requires the fewest iterations to satisfy the stopping criterion, thereby demonstrating its superior computational efficiency and robustness.

6. **Conclusion**

In this paper, we propose a novel double inertial self-adaptive subgradient extragradient method with conjugate gradient direction for solving variational inequality problems involving quasi-monotone and Lipschitz continuous operators in real Hilbert spaces. The proposed algorithm incorporates the technique of double inertial steps combined with a conjugate gradient direction to accelerate the convergence speed. A remarkable feature of our method is the use of a self-adaptive step size strategy, which does not require prior knowledge of the Lipschitz constant of the cost operator. Under some suitable conditions, we establish the weak convergence theorem of the proposed algorithm. Finally, several numerical results are presented to illustrate the feasibility and superior efficiency of our method compared to other algorithms. In future research, we will consider relax the conditional constraints imposed on operator F.

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