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DIRICHLET CHARACTER MODULO 4 ON PARTITIONS

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We extend the Dirichlet character modulo 4 *to integer partitions via an additive statistic on their parts, and introduce several associated partition functions under natural restrictions. Using generating function techniques and classical q-series identities, we establish formulas relating these functions, including representations involving generalized pentagonal numbers and squares. We also obtain an explicit evaluation supported on triangular numbers, together with three distinct infinite families of linear inequalities. These observations further motivate three conjectures concerning infinite families of linear inequalities.*

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1. Introduction.

A *partition* of a non–negative integer n is a way of writing n as a sum of positive integers, called parts, where the order of the summands is irrelevant [1]. Formally, a partition λ of n is a finite non–increasing sequence of positive integers

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell), \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 1,$$

such that

$$|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n.$$

The number $\ell = \ell(\lambda)$ is called the *length* of the partition. The *empty partition* () is the unique partition of 0; including it keeps generating functions and identities uniform without special exceptions. Also define

$$\mathcal{P}(n) = \{\lambda \mid \lambda \text{ is a partition of } n\}.$$

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The classical *Euler partition function* $p(n)$ is defined as

$$p(n) = |\mathcal{P}(n)|,$$

and its generating function is given by

$$\sum_{n=0}^{\infty} p(n)\, q^n = \frac{1}{(q;q)_\infty},$$

where for $|q| < 1$, the q-Pochhammer symbol is defined by

$$(a;q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k).$$

A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is said to be *into distinct parts* if its parts are strictly decreasing. We denote by

$$\mathcal{P}_d(n) = \{\lambda \in \mathcal{P}(n) \mid \lambda_1 > \lambda_2 > \cdots > \lambda_\ell\}$$

the set of all partitions of n into distinct parts.

A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is said to be *into odd parts* if each part is odd. We write

$$\mathcal{P}_o(n) = \{\lambda \in \mathcal{P}(n) \mid \lambda_i \equiv 1 \pmod 2 \text{ for all } i\}$$

for the set of all partitions of n into odd parts.

A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is said to be *into distinct odd parts* if it satisfies both of the above conditions; namely, all parts are odd and strictly decreasing. We define

$$\mathcal{P}_{do}(n) = \{\lambda \in \mathcal{P}_d(n) \mid \lambda_i \equiv 1 \pmod 2 \text{ for all } i\}.$$

Let $E(n)$ be the difference between the number of divisors of n congruent to 1 and 3 modulo 4, i.e.,

$$E(n) := \sum_{d|n} \chi_4(d), \tag{1}$$

where χ_4 is the Dirichlet character modulo 4, namely

$$\chi_4(n) = \begin{cases} \pm 1, & n \equiv \pm 1 \pmod 4, \\ 0, & n \equiv \{0, 2\} \pmod 4. \end{cases}$$

This divisor-difference function was investigated in depth by Glaisher in 1884; see [5] for multiplicativity, structure theorems, and linear recurrences.

In analogy with (1), we extend the Dirichlet character χ_4 from integers to partitions.

Definition 1.1. *For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, we define*

$$X(\lambda) := \sum_{i=1}^{\ell} \chi_4(\lambda_i),$$

that is, the sum of the values of χ_4 evaluated at the parts of λ. For a positive integer n, we further define:

$$S(n) := \sum_{\lambda \in \mathcal{P}(n)} X(\lambda), \qquad S_d^{\pm}(n) := \sum_{\lambda \in \mathcal{P}_d(n)} (\pm 1)^{\ell(\lambda)+1} X(\lambda),$$

$$S_o(n) := \sum_{\lambda \in \mathcal{P}_o(n)} X(\lambda), \qquad S_{do}(n) := \sum_{\lambda \in \mathcal{P}_{do}(n)} X(\lambda).$$

In other words, for a positive integer n, the function $S(n)$ is obtained by considering all partitions λ of n, and for each partition summing the values of the Dirichlet character χ_4 evaluated at its parts. The value $S(n)$ is then the total of these contributions over all partitions of n. The functions $S_d^{\pm}(n)$, $S_o(n)$, and $S_{do}(n)$ are defined analogously, with the summation restricted to partitions into distinct parts, odd parts, and distinct odd parts, respectively. Throughout the paper, we adopt the convention that $S(n) = S_d^{\pm}(n) = S_o(n) = S_{do}(n) = 0$, for $n \leq 0$.

The first result provides explicit representations of the functions $S_d^{-}(n)$, $S_o(n)$, and $S_{do}(n)$ in terms of $S(n)$. In particular, these identities may be interpreted as linear recurrence relations for $S(n)$.

Theorem 1.1. *Let n be a positive integer. Then*

(a) $S_d^{-}(n) = \sum_{i,j \in \mathbb{Z}} (-1)^{i+j} S\big(n - i(3i-1)/2 - j(3j-1)/2\big)$,

(b) $S_o(n) = \sum_{k \in \mathbb{Z}} (-1)^k S\big(n - k(3k-1)\big)$,

(c) $S_{do}(n) = \sum_{k \in \mathbb{Z}} (-1)^k S\big(n - k^2\big)$.

The next result provides alternative expressions for $S_d^{-}(n)$ in terms of $S_d^{+}(n)$, $S_o(n)$ and $S_{do}(n)$, uncovering deeper structural links between partitions with distinct constraints. In particular, these identities may be interpreted as linear recurrence relations for $S_d^{+}(n)$, $S_o(n)$ and $S_{do}(n)$.

Theorem 1.2. *Let n be a positive integer. Then*

(a) $S_d^{-}(n) = \sum_{k \in \mathbb{Z}} (-1)^k S_{do}\big(n - k(3k-1)\big)$,

(b) $S_d^-(n) = \sum_{k\in\mathbb{Z}} (-1)^k \, S_o\big(n-k^2\big)$,

(c) $S_d^-(n) = \sum_{k\in\mathbb{Z}} (-1)^{n+1-k} \, S_d^+\big(n-2k^2\big)$.

By these theorems, we observe that the functions $S(n)$ and $S_{do}(n)$ satisfy the same recurrence relation involving doubled generalized pentagonal numbers, with the only difference arising in the inhomogeneous term. Similarly, the functions $S(n)$ and $S_o(n)$ satisfy a common recurrence relation involving squares, differing only in the inhomogeneous term. As a direct consequence of Theorems 1.1 and 1.2, we obtain the following identity.

Corollary 1.1. *Let n be a positive integer. Then*

$$S_d^-(n) = \sum_{i,j\in\mathbb{Z}} (-1)^{i+j} \, S\big(n-i^2-j(3j-1)\big).$$

Somewhat surprisingly, the two seemingly different expressions

$$\sum_{i,j\in\mathbb{Z}} (-1)^{i+j} \, S\big(n-i(3i-1)/2-j(3j-1)/2\big)$$

and

$$\sum_{i,j\in\mathbb{Z}} (-1)^{i+j} \, S\big(n-i^2-j(3j-1)\big)$$

produce exactly the same recurrence, namely

$$\begin{aligned} &S(n)-2S(n-1)-S(n-2)+2S(n-3)+S(n-4) \\ &\qquad +2S(n-5)-2S(n-6)-2S(n-8)-2S(n-9)+\cdots-S_d^-(n)=0. \end{aligned}$$

The following result provides new linear recurrences for $S_d^+(n)$ and $S_{do}(n)$ involving double and quadrupled generalized pentagonal numbers, which are homogeneous except at triangular numbers.

Theorem 1.3. *Let n be a positive integer. Then*

(a) $\displaystyle\sum_{k\in\mathbb{Z}} (-1)^k \, S_{do}\big(n-2k(3k-1)\big) = \begin{cases} (-1)^{\lfloor m/2\rfloor} \lceil m/2\rceil, & \text{if } n=\frac{m(m+1)}{2},\ m\in\mathbb{N}_0, \\ 0, & \text{otherwise;} \end{cases}$

(b) $\displaystyle\sum_{k\in\mathbb{Z}} (-1)^k \, S_d^+\big(n-k(3k-1)\big) = \begin{cases} (-1)^{m-1} \lceil m/2\rceil, & \text{if } n=\frac{m(m+1)}{2},\ m\in\mathbb{N}_0, \\ 0, & \text{otherwise.} \end{cases}$

The structure of the paper is as follows. In Section 2, we establish the proof of Theorem 1.1 via generating function techniques. In Section 3, we provide a proof of Theorem 1.2. Section 4 is devoted to the proof of Theorem

1.3. In Section 5, we introduce three infinite families of linear inequalities obtained from truncations of classical q-series identities. In the final section, we propose three conjectures concerning infinite families of linear inequalities.

2. Proof of Theorem 1.1

A standard argument shows that the generating function of $E(n)$ is given by the Lambert series

$$\sum_{n=1}^{\infty} E(n)\, q^n = \sum_{n=1}^{\infty} \frac{\chi_4(n)\, q^n}{1-q^n}, \qquad |q|<1.$$

By [10, Proposition 2.4], this series admits the factorization

$$\sum_{n=1}^{\infty} \frac{\chi_4(n)\, q^n}{1-q^n} = (q;q)_\infty \sum_{n=1}^{\infty} S(n)\, q^n. \tag{2}$$

On the other hand, applying [8, Theorem 1.2] with $a_n = \chi_4(n)$ yields

$$\sum_{n=1}^{\infty} \frac{\chi_4(n)\, q^n}{1-q^n} = \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} S_d^-(n)\, q^n. \tag{3}$$

Combining (2) and (3), we obtain

$$\sum_{n=1}^{\infty} S_d^-(n)\, q^n = (q;q)_\infty^2 \sum_{n=1}^{\infty} S(n)\, q^n.$$

Invoking Euler's pentagonal number theorem,

$$(q;q)_\infty = \sum_{n\in\mathbb{Z}} (-1)^n q^{n(3n-1)/2},$$

we deduce

$$\begin{aligned}
\sum_{n=1}^{\infty} S_d^-(n)\, q^n &= \left(\sum_{n\in\mathbb{Z}} (-1)^n q^{n(3n-1)/2}\right)^2 \left(\sum_{n=1}^{\infty} S(n)\, q^n\right) \\
&= \left(\sum_{i,j\in\mathbb{Z}} (-1)^{i+j} q^{i(3i-1)/2+j(3j-1)/2}\right) \left(\sum_{n=1}^{\infty} S(n)\, q^n\right).
\end{aligned}$$

Extracting coefficients yields

$$S_d^-(n) = \sum_{i,j\in\mathbb{Z}} (-1)^{i+j}\, S\big(n - i(3i-1)/2 - j(3j-1)/2\big),$$

which proves the first identity.

Next, observing that

$$\sum_{n=1}^{\infty} \frac{\chi_4(n)\, q^n}{1-q^n} = \sum_{n=1}^{\infty} \frac{\chi_4(2n-1)\, q^{2n-1}}{1-q^{2n-1}}, \tag{4}$$

we may apply again [10, Proposition 2.4] to obtain

$$\sum_{n=1}^{\infty} \frac{\chi_4(n)\, q^n}{1-q^n} = (q;q^2)_\infty \sum_{n=1}^{\infty} S_o(n)\, q^n. \tag{5}$$

Combining (2) and (5), we deduce

$$\sum_{n=1}^{\infty} S_o(n)\, q^n = (q^2;q^2)_\infty \sum_{n=1}^{\infty} S(n)\, q^n.$$

Using again Euler's pentagonal number theorem, we obtain

$$\sum_{n=1}^{\infty} S_o(n)\, q^n = \left(\sum_{n\in\mathbb{Z}} (-1)^n q^{n(3n-1)} \right) \left(\sum_{n=1}^{\infty} S(n)\, q^n \right).$$

Comparing coefficients gives

$$S_o(n) = \sum_{k\in\mathbb{Z}} (-1)^k\, S\big(n-k(3k-1)\big),$$

which proves the second identity.

Finally, taking into account (4) and [10, Proposition 2.3], we have

$$\sum_{n=1}^{\infty} \frac{\chi_4(n)\, q^n}{1-q^n} = \frac{1}{(q;q^2)_\infty} \sum_{n=1}^{\infty} S_{do}(n)\, q^n. \tag{6}$$

Combining (2) and (6), we obtain

$$\sum_{n=1}^{\infty} S_{do}(n)\, q^n = \frac{(q;q)_\infty}{(-q;q)_\infty} \sum_{n=1}^{\infty} S(n)\, q^n.$$

Using Jacobi's theta identity [1, p. 23, eq. (2.2.12)],

$$\frac{(q;q)_\infty}{(-q;q)_\infty} = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2},$$

we deduce

$$\sum_{n=1}^{\infty} S_{do}(n)\, q^n = \left(\sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} \right) \left(\sum_{n=1}^{\infty} S(n)\, q^n \right).$$

Extracting coefficients yields

$$S_{do}(n) = \sum_{k\in\mathbb{Z}} (-1)^k\, S\big(n-k^2\big),$$

which completes the proof.

3. Proof of Theorem 1.2

By combining (3) and (6), we obtain

$$\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty} S_d^-(n)\, q^n = \frac{1}{(q;q^2)_\infty}\sum_{n=1}^{\infty} S_{do}(n)\, q^n.$$

It follows that

$$\begin{aligned}\sum_{n=1}^{\infty} S_d^-(n)\, q^n &= \frac{1}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty} S_{do}(n)\, q^n \\ &= \left(\sum_{n\in\mathbb{Z}}(-1)^n\, q^{n(3n-1)}\right)\left(\sum_{n=1}^{\infty} S_{do}(n)\, q^n\right).\end{aligned}$$

Extracting coefficients yields

$$S_d^-(n) = \sum_{k\in\mathbb{Z}}(-1)^k\, S_{do}\big(n-k(3k-1)\big),$$

which proves the first identity.

Similarly, combining (3) and (5), we obtain

$$\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty} S_d^-(n)\, q^n = (q;q^2)_\infty\sum_{n=1}^{\infty} S_o(n)\, q^n,$$

and hence

$$\begin{aligned}\sum_{n=1}^{\infty} S_d^-(n)\, q^n &= \frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n=1}^{\infty} S_o(n)\, q^n \\ &= \left(\sum_{n\in\mathbb{Z}}(-1)^n\, q^{n^2}\right)\left(\sum_{n=1}^{\infty} S_o(n)\, q^n\right).\end{aligned}$$

Comparing coefficients gives

$$S_d^-(n) = \sum_{k\in\mathbb{Z}}(-1)^k\, S_o\big(n-k^2\big),$$

which completes the proof of the second identity.

Applying [8, Theorem 1.2], with $a_n=\chi_4(n)$ yields

$$\sum_{n=1}^{\infty}\frac{\chi_4(n)\, q^n}{1+q^n} = \frac{1}{(-q;q)_\infty}\sum_{n=1}^{\infty} S_d^+(n)\, q^n. \tag{7}$$

On the other hand, we have

$$\sum_{n=1}^{\infty} \frac{\chi_4(n)\,(-q)^n}{1+(-q)^n} = -\sum_{n=1}^{\infty} \frac{\chi_4(n)\,q^n}{1-q^n}.$$

Thus, considering (3) we obtain

$$\frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} S_d^-(n)\,q^n = \frac{1}{(q;-q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1}\,S_d^+(n)\,q^n,$$

and hence

$$\begin{aligned}
\sum_{n=1}^{\infty} S_d^-(n)\,q^n &= \frac{(q;q)_\infty}{(q;-q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1}\,S_d^+(n)\,q^n \\
&= \frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1}\,S_d^+(n)\,q^n \\
&= \left(\sum_{n\in\mathbb{Z}} (-1)^n\,q^{2n^2} \right) \left(\sum_{n=1}^{\infty} (-1)^{n-1}\,S_d^+(n)\,q^n \right).
\end{aligned}$$

Comparing coefficients gives

$$S_d^-(n) = \sum_{k\in\mathbb{Z}} (-1)^{n-1-k}\,S_d^+\big(n-2k^2\big),$$

which completes the proof.

4. **Proof of Theorem 1.3**

We begin with the following factorization for the generating function of $E(n)$, recently established in [9]:

$$\sum_{n=1}^{\infty} \frac{\chi_4(n)\,q^n}{1-q^n} = \frac{1}{(q;q^2)_\infty (q^4;q^4)_\infty} \sum_{n=1}^{\infty} (-1)^{\lfloor n/2 \rfloor} \lceil n/2 \rceil\, q^{n(n+1)/2}. \tag{8}$$

We combine (8) with (6). Substituting the expression for the generating function of $S_{do}(n)$, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} (-1)^{\lfloor n/2 \rfloor} \lceil n/2 \rceil\, q^{n(n+1)/2} &= (q^4;q^4)_\infty \sum_{n=1}^{\infty} S_{do}(n)\,q^n \\
&= \left(\sum_{k\in\mathbb{Z}} (-1)^k q^{2k(3k-1)} \right) \left(\sum_{n=1}^{\infty} S_{do}(n)\,q^n \right),
\end{aligned}$$

where, in the second step, we used Euler's pentagonal number theorem.

We now expand the product on the right-hand side. Collecting coefficients of q^n, we deduce

$$\sum_{n=1}^{\infty}(-1)^{\lfloor n/2\rfloor}\,\lceil n/2\rceil\,q^{n(n+1)/2}=\sum_{n=1}^{\infty}q^n\sum_{k\in\mathbb{Z}}(-1)^k\,S_{do}\big(n-2k(3k-1)\big),$$

which establishes the first identity.

To prove the second identity, we return to (8) and replace q by $-q$. This yields

$$\sum_{n=1}^{\infty}\frac{\chi_4(n)\,q^n}{1+q^n}=\frac{1}{(-q;q^2)_\infty(q^4;q^4)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}\,\lceil n/2\rceil\,q^{n(n+1)/2}.$$

On the other hand, according to by (7), we can write

$$\frac{1}{(-q;q)_\infty}\sum_{n=1}^{\infty}S_d^{+}(n)\,q^n=\frac{1}{(-q;q^2)_\infty(q^4;q^4)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}\,\lceil n/2\rceil\,q^{n(n+1)/2}.$$

and hence

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1}\,\lceil n/2\rceil\,q^{n(n+1)/2}&=\frac{(-q;q^2)_\infty(q^4;q^4)_\infty}{(-q;q)_\infty}\sum_{n=1}^{\infty}S_d^{+}(n)\,q^n\\&=(q^2;q^2)_\infty\sum_{n=1}^{\infty}S_d^{+}(n)\,q^n.\end{aligned}$$

Finally, applying Euler's pentagonal number theorem once again, we rewrite

$$\sum_{n=1}^{\infty}(-1)^{n-1}\,\lceil n/2\rceil\,q^{n(n+1)/2}=\left(\sum_{k\in\mathbb{Z}}(-1)^k\,q^{k(3k-1)}\right)\left(\sum_{n=1}^{\infty}S_d^{+}(n)\,q^n\right).$$

Expanding the product and collecting coefficients of q^n, we conclude that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\,\lceil n/2\rceil\,q^{n(n+1)/2}=\sum_{n=1}^{\infty}q^n\sum_{k\in\mathbb{Z}}(-1)^k\,S_d^{+}\big(n-k(3k-1)\big),$$

which completes the proof.

5. **Infinite families of linear inequalities**

In this section, we derive three infinite families of linear inequalities involving the functions $S(n)$, $S_o(n)$, $S_d^{-}(n)$, and $S_{do}(n)$. These inequalities arise from truncations of classical q-series identities, namely Euler's pentagonal number theorem and Jacobi-type theta identities. Our approach relies on the principle that truncated expansions of infinite products yield inequalities with controlled error terms; see, for example, [2, 3, 4, 7].

We begin with a family of inequalities obtained from truncations of Euler's pentagonal number theorem.

Theorem 5.1. *Let k be a positive integer. For $n \geqslant 0$,*

$$(-1)^k \left(S_o(n) - \sum_{j=1-k}^{k} (-1)^j \, S\big(n - j(3j-1)\big) \right) \geq 0,$$

with strict inequality if $n > k(3k+1)$.

Proof. By the truncated pentagonal number theorem [2], we have

$$[q^n]\,(-1)^k \left(1 - \frac{1}{(q;q)_\infty} \sum_{j=1-k}^{k} (-1)^j q^{j(3j-1)/2} \right) \geq 0,$$

with strict inequality for $n \geq k(3k+1)/2$.

Let

$$r_2(n) = \#\{(a,b) \in \mathbb{Z}^2 : \; a^2 + b^2 = n\}.$$

By Jacobi's classical theorem (see [6, Theorem 278]),

$$r_2(n) = 4 \sum_{d|n} \chi_4(d) = 4E(n),$$

and hence $E(n) \geq 0$ for all $n \geq 1$.

It follows that

$$\sum_{n=1}^{\infty} S_o(n)\, q^n = \frac{1}{(q;q^2)_\infty} \sum_{n=1}^{\infty} E(n)\, q^n$$

is a power series with nonnegative coefficients.

Multiplying the two series, we obtain

$$[q^n]\,(-1)^k \left(1 - \frac{1}{(q^2;q^2)_\infty} \sum_{j=1-k}^{k} (-1)^j q^{j(3j-1)} \right) \sum_{n=1}^{\infty} S_o(n)\, q^n \geq 0,$$

with strict inequality if $n > k(3k+1)$.

Expanding the product, we get

$$[q^n]\,(-1)^k \left(\sum_{n=1}^{\infty} S_o(n)\, q^n - \left(\sum_{n=1}^{\infty} S(n)\, q^n \right) \left(\sum_{j=1-k}^{k} (-1)^j q^{j(3j-1)} \right) \right) \geq 0,$$

which is equivalent to

$$[q^n]\,(-1)^k \left(\sum_{n=1}^{\infty} S_o(n)\, q^n - \sum_{n=1}^{\infty} q^n \sum_{j=1-k}^{k} (-1)^j S\big(n - j(3j-1)\big) \right) \geq 0.$$

Comparing coefficients of q^n, we obtain the desired inequality. □

We now derive a second family of inequalities, based on truncations of theta series involving squares.

Theorem 5.2. *Let k be a positive integer. For $n \geqslant 0$,*

$$(-1)^{k-1}\left(S_{do}(n) - \sum_{j=-k}^{k}(-1)^j\, S\big(n-j^2\big)\right) \geq 0,$$

with strict inequality if $n > (k+1)^2$.

Proof. According to [4], for any integer $k > 0$ and $n \geq 0$, we have

$$[q^n]\,(-1)^{k-1}\left((q;q^2)_\infty - \frac{1}{(q;q)_\infty}\sum_{j=-k}^{k}(-1)^j q^{j^2}\right) \geq 0,$$

with strict inequality for $n \geq (k+1)^2$.

Using again the fact that $E(n) \geq 0$, it follows that

$$[q^n]\,(-1)^{k-1}\left((q;q^2)_\infty - \frac{1}{(q;q)_\infty}\sum_{j=-k}^{k}(-1)^j q^{j^2}\right)\sum_{n=1}^{\infty}\frac{\chi_4(n)\,q^n}{1-q^n} \geq 0,$$

with strict inequality if $n > (k+1)^2$.

Expanding the product, we obtain

$$[q^n]\,(-1)^{k-1}\left(\sum_{n=1}^{\infty} S_{do}(n)\,q^n - \left(\sum_{n=1}^{\infty} S(n)\,q^n\right)\left(\sum_{j=-k}^{k}(-1)^j q^{j^2}\right)\right) \geq 0,$$

which is equivalent to

$$[q^n]\,(-1)^{k-1}\left(\sum_{n=1}^{\infty} S_{do}(n)\,q^n - \sum_{n=1}^{\infty} q^n \sum_{j=-k}^{k}(-1)^j S\big(n-j^2\big)\right) \geq 0.$$

Comparing coefficients of q^n yields the claim. □

Finally, we obtain a third family of inequalities linking the functions $S_d(n)$ and $S_o(n)$.

Theorem 5.3. *Let k be a positive integer. For $n \geqslant 0$,*

$$(-1)^{k-1}\left(S_d^{-}(n) - \sum_{j=-k}^{k}(-1)^j\, S_o\big(n-j^2\big)\right) \geq 0,$$

with strict inequality if $n > (k+1)^2$.

Proof. Let $Q(n)=|\mathcal{P}_o(n)|$. By Euler's classical theorem, we have

$$|\mathcal{P}_d(n)|=|\mathcal{P}_o(n)|,$$

and hence

$$\frac{1}{(q;q^2)_\infty}=\sum_{n=0}^{\infty}Q(n)\,q^n.$$

Using Jacobi's identity

$$\sum_{j\in\mathbb{Z}}(-1)^j q^{j^2}=(q;q)_\infty(q;q^2)_\infty,$$

we may write

$$\begin{aligned}(q;q)_\infty-\frac{1}{(q;q^2)_\infty}\sum_{j=-k}^{k}(-1)^j q^{j^2}&=\frac{1}{(q;q^2)_\infty}\left(\sum_{j\in\mathbb{Z}}(-1)^j q^{j^2}-\sum_{j=-k}^{k}(-1)^j q^{j^2}\right)\\&=\frac{2}{(q;q^2)_\infty}\sum_{j=k+1}^{\infty}(-1)^j q^{j^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}&[q^n]\,(-1)^{k-1}\left((q;q)_\infty-\frac{1}{(q;q^2)_\infty}\sum_{j=-k}^{k}(-1)^j q^{j^2}\right)\\&\quad=[q^n]\,(-1)^{k-1}\,2\left(\sum_{m=0}^{\infty}Q(m)\,q^m\right)\left(\sum_{j=k+1}^{\infty}(-1)^j q^{j^2}\right)\\&\quad=(-1)^{k-1}\,2\sum_{j=k+1}^{\infty}(-1)^j Q(n-j^2).\end{aligned}$$

Since $Q(n)$ is a nondecreasing function of n, it follows that the right-hand side is nonnegative. Multiplying by the Lambert series

$$\sum_{n=1}^{\infty}\frac{\chi_4(n)\,q^n}{1-q^n}=\sum_{n=1}^{\infty}E(n)\,q^n,$$

and using again the fact that $E(n)\geq 0$, we obtain

$$[q^n]\,(-1)^{k-1}\left((q;q)_\infty-\frac{1}{(q;q^2)_\infty}\sum_{j=-k}^{k}(-1)^j q^{j^2}\right)\sum_{n=1}^{\infty}\frac{\chi_4(n)\,q^n}{1-q^n}\geq 0,$$

with strict inequality if $n>(k+1)^2$.

Expanding the product, we obtain

$$[q^n]\,(-1)^{k-1}\left(\sum_{n=1}^{\infty} S_d^-(n)\,q^n - \left(\sum_{n=1}^{\infty} S_o(n)\,q^n\right)\left(\sum_{j=-k}^{k}(-1)^j q^{j^2}\right)\right) \geq 0,$$

which is equivalent to

$$[q^n]\,(-1)^{k-1}\left(\sum_{n=1}^{\infty} S_d^-(n)\,q^n - \sum_{n=1}^{\infty} q^n \sum_{j=-k}^{k}(-1)^j S_o\big(n-j^2\big)\right) \geq 0.$$

Comparing coefficients of q^n yields the claim. □

6. Concluding remarks and open problems

In this paper, we introduced a partition-theoretic extension of the Dirichlet character modulo 4 and investigated the associated functions $S(n)$, $S_d^{\pm}(n)$, $S_o(n)$, and $S_{do}(n)$. Using generating function techniques and classical q-series identities, we established several formulas linking these functions, including representations involving generalized pentagonal numbers and squares.

A notable feature of our results is that these functions satisfy closely related recurrence relations, differing only in their inhomogeneous terms. In particular, we obtained a new recurrence for $S_{do}(n)$ involving quadrupled generalized pentagonal numbers, which is homogeneous except at triangular numbers. We also derived three infinite families of linear inequalities arising from truncations of classical theta series.

The identities and structural similarities observed throughout the paper suggest the existence of deeper combinatorial and analytic connections between these partition functions. Motivated by these findings, we formulate three conjectures relating $S_d^{\pm}(n)$ and $S_{do}(n)$, each exhibiting behavior similar to that of the previously established inequalities.

Conjecture 6.1. *Let k be a positive integer and let $t \in \{1,3\}$. Suppose that $k \equiv t-1$ or $t \pmod 4$, and that $n \equiv t \pmod 4$. Then*

$$(-1)^k\left(S_d^-(n) - \sum_{j=1-k}^{k}(-1)^j\,S_{do}\big(n-j(3j-1)\big)\right) \geq 0.$$

Conjecture 6.2. *Let k be a positive integer. For $n \geq 0$,*

$$(-1)^{k-1}\left((-1)^{n-1}\,S_d^-(n) - \sum_{j=-k}^{k}(-1)^j\,S_d^+\big(n-2j^2\big)\right) \geq 0.$$

Conjecture 6.3. *Let k be a positive integer. For $n \geq 0$,*

$$(-1)^{k-1}\left(\rho(n) - \sum_{j=1-k}^{k} (-1)^j\, S_d^+\big(n - j(3j-1)\big)\right) \geq 0,$$

where

$$\rho(n) = \begin{cases} (-1)^{m-1}\,\lceil m/2 \rceil, & \text{if } n = \frac{m(m+1)}{2},\ m \in \mathbb{N}_0, \\ 0, & \text{otherwise.} \end{cases}$$

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