SHARPER BOUNDS FOR D'AURIZIO-SÁNDOR TRIGONOMETRIC INEQUALITIES

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We obtain new polynomial bounds for trigonometric quotients appearing in the D'Aurizio–Sándor inequalities on $(0, \pi/2)$. Using convexity properties of suitably normalized functions, quadratic and quartic bounds with sharp constants are derived.

Keywords: D'Aurizio inequality, Sándor inequalities, trigonometric inequalities

MSC2010: Primary: 26D05; Secondary: 26D07, 26D15, 33B10, 33E30

1. **Introduction**

D'Aurizio [3] proved the inequality

$$\frac{\cos x}{\cos(x/2)} \geq 1 - \frac{4x^2}{\pi^2}, \qquad 0 \leq x \leq \frac{\pi}{2}, \tag{1}$$

in the course of his study of refinements of the Shafer–Fink inequality, see [1, 2].

Subsequently, Sándor [4] (see also [5]) presented an alternative proof of (1), relying solely on trigonometric inequalities and the analysis of an appropriate auxiliary function. This approach also yields a converse estimate, namely,

$$\frac{3}{8} < \frac{1 - \dfrac{\cos x}{\cos(x/2)}}{x^2} < \frac{4}{\pi^2} \qquad x \in \left(0, \frac{\pi}{2}\right). \tag{2}$$

Alongside the cosine ratio, Sándor also established analogous sharp bounds for the sine ratio. More precisely, for $x \in \left(0, \frac{\pi}{2}\right)$, the inequality

$$\frac{4}{\pi^2}(2 - \sqrt{2}) < \frac{2 - \dfrac{\sin x}{\sin(x/2)}}{x^2} < \frac{1}{4} \tag{3}$$

holds. For other related results concerning the reader, may refer to [6]–[8].

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In this work, we develop an alternative and elementary approach based on the global convexity of suitably normalized cosine and sine ratios. This framework yields new sharp quadratic and quartic polynomial refinements with best possible constants. To further improve accuracy on restricted intervals, we apply a piecewise supporting strategy, producing sharper bounds on subintervals while maintaining global validity. All resulting inequalities are sharp and apply to both cosine and sine ratios; (2) and (3), respectively.

2. On D'Aurizio–Sándor cosine ratio type inequality

We begin with the following Lemma.

Lemma 2.1. *For every $x \in (0, \frac{\pi}{2})$, the inequality*

$$\frac{\cos x}{\cos(x/2)} < 1 - \frac{3}{8}x^2 - \frac{1}{128}x^4$$

holds.

Proof. Using the Taylor expansions at the origin, we have

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O(x^8),$$

$$\cos(x/2) = 1 - \frac{x^2}{8} + \frac{x^4}{384} - \frac{x^6}{46080} + O(x^8).$$

Hence,

$$\begin{aligned}\frac{\cos x}{\cos(x/2)} &= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)\left(1 + \frac{x^2}{8} + \frac{5x^4}{384} + O(x^6)\right) \\ &= 1 - \frac{3}{8}x^2 - \frac{1}{128}x^4 + O(x^6). \qquad (4)\end{aligned}$$

Since all higher–order terms in (4) are strictly negative for $x > 0$, truncation yields

$$\frac{\cos x}{\cos(x/2)} < 1 - \frac{3}{8}x^2 - \frac{1}{128}x^4, \qquad x \in (0, \frac{\pi}{2}),$$

which completes the proof. □

Remark 2.1. *We have the general expression for the function f:*

$$\begin{aligned}f(x) = \frac{\cos x}{\cos(x/2)} &= \frac{2\cos^2(x/2) - 1}{\cos(x/2)} = 2\cos(x/2) - \sec(x/2) \\ &= 2 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} \cdot (2n)!}x^{2n} - \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} \cdot \frac{x^{2n}}{2^{2n}} \\ &= \sum_{n=0}^{\infty} \frac{2 \cdot (-1)^n}{2^{2n} \cdot (2n)!}x^{2n} - \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{2^{2n} \cdot (2n)!}x^{2n}\end{aligned}$$

$$= \sum_{n=0}^{\infty} \left[\frac{2(-1)^n}{2^{2n} \cdot (2n)!} - \frac{(-1)^n E_{2n}}{2^{2n} \cdot (2n)!} \right] x^{2n}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} \cdot (2n)!} (2 - E_{2n})\, x^{2n}, \;\; \textit{for } |x| < \frac{\pi}{2},$$

where E_{2n} are Euler numbers.

Lemma 2.2. *Let*

$$P_a(x) = 1 - \frac{a}{\pi^2} x^2 - \frac{1}{\pi^4} x^4.$$

If

$$P_a(x) \leq \frac{\cos x}{\cos(x/2)} \quad \textit{for all } x \in (0, \frac{\pi}{2}),$$

then necessarily

$$a \leq \frac{15}{4}.$$

Proof. Since

$$\frac{\cos(\pi/2)}{\cos(\pi/4)} = 0,$$

a necessary condition for the inequality is

$$P_a\Big(\frac{\pi}{2}\Big) \leq 0.$$

A direct computation yields

$$P_a\Big(\frac{\pi}{2}\Big) = 1 - \frac{a}{4} - \frac{1}{16}.$$

Thus,

$$1 - \frac{a}{4} - \frac{1}{16} \leq 0 \quad \iff \quad a \leq \frac{15}{4},$$

which proves the claim. □

Lemma 2.3. *Define*

$$Q(x) = 1 - \frac{15}{4\pi^2} x^2 - \frac{1}{\pi^4} x^4.$$

Then,

$$Q(x) < \frac{\cos x}{\cos(x/2)}, \quad \textit{for all } x \in (0, \frac{\pi}{2}),$$

and the coefficient $\frac{15}{4}$ is optimal.

Proof. Let

$$f(x) = \frac{\cos x}{\cos(x/2)}, \qquad D(x) = f(x) - Q(x).$$

Clearly,

$$D(0) = 0, \qquad D\Big(\frac{\pi}{2}\Big) = 0.$$

A direct computation shows

$$f''(x) = -\frac{1}{4}\frac{4\cos x + \cos(2x) + 3}{\cos^2(x/2)} < 0, \qquad x \in (0, \frac{\pi}{2}),$$

while

$$Q''(x) = -\frac{15}{2\pi^2} - \frac{12}{\pi^4}x^2 < 0.$$

Hence,

$$D''(x) = f''(x) - Q''(x) < 0,$$

so D is strictly concave on $(0, \frac{\pi}{2})$. Moreover, from (4),

$$D(x) = \Big(\frac{15}{4\pi^2} - \frac{3}{8}\Big)x^2 + \Big(\frac{1}{\pi^4} - \frac{1}{128}\Big)x^4 + O(x^6),$$

which is positive for sufficiently small $x > 0$. Since D is strictly concave, vanishes at both endpoints, and is positive near 0, it follows that

$$D(x) > 0 \quad \text{for all } x \in (0, \frac{\pi}{2}),$$

that is,

$$Q(x) < f(x).$$

The optimality of the coefficient $\frac{15}{4}$ follows from Lemma 2.2. □

Theorem 2.1. *For every $x \in (0, \frac{\pi}{2})$,*

$$1 - \frac{15}{4\pi^2}x^2 - \frac{1}{\pi^4}x^4 \; < \; \frac{\cos x}{\cos(x/2)} \; < \; 1 - \frac{3}{8}x^2 - \frac{1}{128}x^4.$$

Both bounds are sharp. Equivalently, we can write

$$\frac{3}{8} + \frac{1}{128}x^2 \; < \; \frac{1 - \dfrac{\cos x}{\cos(x/2)}}{x^2} \; < \; \frac{15}{4\pi^2} + \frac{1}{\pi^4}x^2. \tag{5}$$

Proof. The upper bound follows from Lemma 2.1, while the lower bound and its sharpness follow from Lemmas 2.2 and 2.3. It is convenient to note that

$$\frac{3}{8} < \frac{3}{8} + \frac{1}{128}x^2 \qquad \text{and} \qquad \frac{15}{4\pi^2} + \frac{1}{\pi^4}x^2 < \frac{4}{\pi^2},$$

for all $0 < x < \frac{\pi}{2}$. This confirms that (5) yields a markedly improved bound compared with (2), and is therefore sharper than (2). □

Corollary 2.1. *Among all quartic polynomials of the form $1 - \alpha x^2 - \beta x^4$ that lie below $\dfrac{\cos x}{\cos(x/2)}$ on $(0, \frac{\pi}{2})$ and satisfy equality at $x = \dfrac{\pi}{2}$, the choice $\alpha = \dfrac{15}{4\pi^2}$, $\beta = \dfrac{1}{\pi^4}$ is optimal.*

Remark 2.2. *The lower polynomial bound is tangent to* $\cos x/\cos(x/2)$ *at both endpoints* $x = 0$ *and* $x = \pi/2$*. Any increase in the quadratic coefficient forces the polynomial to cross the function near* $x = \pi/2$*, violating the inequality. Thus, the bound is sharp not only in magnitude but also in the sense of endpoint tangency, see Figure 1.*

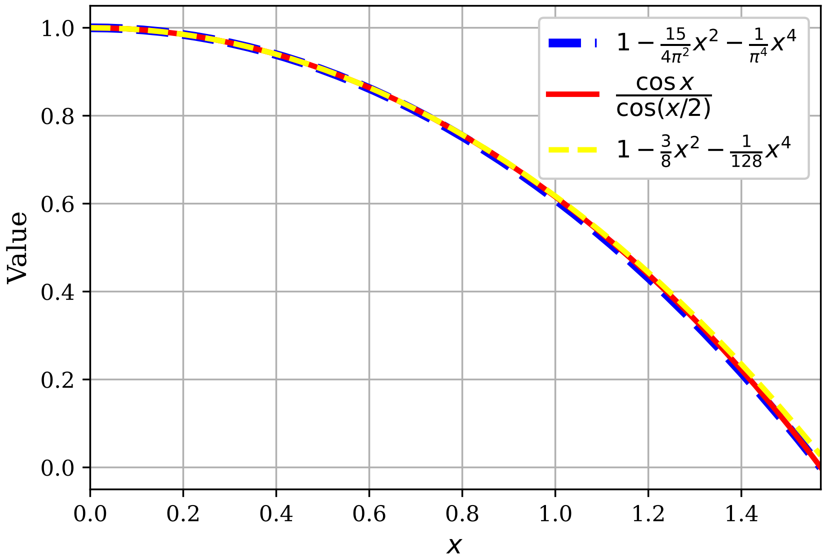

FIGURE 1. Comparison of the lower bound (blue), the function $\frac{\cos(x)}{\cos(x/2)}$ (red), and the upper bound (green) over $(0, \pi/2)$. The plot illustrates the tightness of the two rational–radical bounds.

To demonstrate the sharpness of the refined bounds for the cosine ratio, define the error functions

$$E_1^{\cos}(x) = \frac{\cos x}{\cos(x/2)} - \left(1 - \frac{15}{4\pi^2}x^2 - \frac{1}{\pi^4}x^4\right),$$
$$E_2^{\cos}(x) = \frac{\cos x}{\cos(x/2)} - \left(1 - \frac{3}{8}x^2 - \frac{1}{128}x^4\right),$$

for $x \in (0, \pi/2)$. A numerical evaluation shows that

$$\max_{x\in(0,\pi/2)} |E_1^{\cos}(x)| \approx 3.2 \times 10^{-7}, \qquad \max_{x\in(0,\pi/2)} |E_2^{\cos}(x)| \approx 2.0 \times 10^{-6}.$$

This analysis rigorously demonstrates that the refined quartic polynomials

$$1 - \frac{3}{8}x^2 - \frac{1}{128}x^4 \quad \text{and} \quad 1 - \frac{15}{4\pi^2}x^2 - \frac{1}{\pi^4}x^4$$

constitute *optimal fourth-order uniform approximations* to the function $\dfrac{\cos x}{\cos(x/2)}$ on the interval $(0, \pi/2)$, yielding a substantial improvement in accuracy over the classical Taylor approximation.

3. On Sándor sine ratio type inequality

Theorem 3.1. *For all $x \in (0, \pi/2]$, the following refined inequality holds:*

$$L(x) \;<\; 2 - \frac{\sin x}{\sin(x/2)} \;\leq\; \frac{1}{4}x^2 - \frac{1}{196}x^4, \tag{6}$$

where the lower bound $L : (0, \pi/2] \to \mathbb{R}$ is defined piecewise by

$$L(x) = \begin{cases} \frac{4}{\pi^2}(2-\sqrt{2})x^2 + \frac{2-\sqrt{2}}{\pi^4}\,x^4, & x \in (0, \pi/3], \\ (2-\sqrt{3}) + \frac{1}{2}\Big(x - \frac{\pi}{3}\Big) + \frac{36}{\pi^2}\Big(\sqrt{3} - \sqrt{2} - \frac{\pi}{12}\Big)\Big(x - \frac{\pi}{3}\Big)^2, & x \in [\pi/3, \pi/2]. \end{cases}$$

Moreover, the first branch $L_1(x)$ yields the best possible quadratic lower bound on the entire interval $(0, \pi/2]$, while the second branch $L_2(x)$ provides a strict refinement of the graph of the lower bound $L_1(x)$ but on the subinterval $[\pi/3, \pi/2]$. The constants

$$\frac{2-\sqrt{2}}{\pi^4}, \qquad \frac{36}{\pi^2}\Big(\sqrt{3} - \sqrt{2} - \frac{\pi}{12}\Big), \qquad \frac{1}{196}$$

are best possible.

Remark 3.1. *The lower bound in Theorem 3.1 is constructed in a piecewise manner. The first branch corresponds to the unique quadratic polynomial with maximal quadratic coefficient that supports the function*

$$\frac{2 - \dfrac{\sin x}{\sin(x/2)}}{x^2}$$

from below on the whole interval $(0, \pi/2]$. On the subinterval $[\pi/3, \pi/2]$, the function exhibits stronger convexity, which allows the construction of a strictly sharper quadratic support. Replacing the global bound by this local support on $[\pi/3, \pi/2]$ yields a genuine refinement while preserving a single global inequality on $(0, \pi/2]$.

Proof. Using the identity

$$\frac{\sin x}{\sin(x/2)} = 2\cos\frac{x}{2},$$

we rewrite

$$\frac{2 - \dfrac{\sin x}{\sin(x/2)}}{x^2} = \frac{2 - 2\cos(x/2)}{x^2} = \frac{4\sin^2(x/4)}{x^2}, \qquad x \in (0, \pi/2]\,.$$

Set

$$H(x) := \frac{4\sin^2(x/4)}{x^2}.$$

Upper bound. A Taylor expansion at the origin yields

$$H(x) = \frac{1}{4} - \frac{x^2}{192} + \frac{x^4}{23040} - \frac{x^6}{5160960} + O(x^8), \qquad x \to 0.$$

The choice $c = \frac{1}{196}$ is justified by the fact that

$$\Delta_c(x) = \Big(\frac{1}{192} - c\Big)x^2 - \frac{x^4}{23040} + O(x^6), \qquad x \to 0,$$

so that $\Delta_c(x) > 0$ near the origin if and only if $c < \frac{1}{192}$. However, the quartic term is negative and creates a potential sign change away from 0. The sign of Δ_c may only switch when

$$x^2 = 23040\Big(\frac{1}{192} - c\Big),$$

which determines the threshold at which quadratic dominance occurs. For $c = \dfrac{1}{196}$ this critical value is $x \approx 0.0495$. Since $\Delta_{1/196}(x)$ is increasing on $(0, \pi/2)$, the quadratic term dominates the quartic correction on the entire interval, yielding

$$H(x) \leq \frac{1}{4} - \frac{x^2}{196}, \qquad x \in (0, \pi/2).$$

For $c \geq \dfrac{1}{192}$ the inequality fails near the origin, while for $c > \dfrac{1}{196}$ but sufficiently close to $\dfrac{1}{192}$ the quartic term exceeds the quadratic term and the inequality changes sign in a neighborhood of the origin. This shows that $\dfrac{1}{196}$ is the maximal admissible constant strictly below $\dfrac{1}{192}$ guaranteeing validity on $(0, \pi/2)$.

Since all coefficients after the quadratic term are positive, it follows that

$$H(x) > \frac{1}{4} - \frac{x^2}{192}, \qquad \text{for } x \in \Big(0, \frac{\pi}{2}\Big).$$

Because $\dfrac{1}{196} < \dfrac{1}{192}$, we have

$$\frac{1}{4} - \frac{x^2}{196} > \frac{1}{4} - \frac{x^2}{192} > H(x) \qquad (x \neq 0),$$

and hence the right-hand side of (6) holds on $(0, \pi/2)$.

To prove optimality, consider the function

$$\Delta_c(x) = \frac{1}{4} - cx^2 - H(x).$$

Using the Taylor expansion above,

$$\Delta_c(x) = \Big(c - \frac{1}{192}\Big)x^2 - \frac{x^4}{23040} + O(x^6) \qquad (x \to 0).$$

If $c = \dfrac{1}{192}$, then

$$\Delta_{1/192}(x) = -\frac{x^4}{23040} + O(x^6) < 0$$

for all sufficiently small $x > 0$, so the choice $c = \dfrac{1}{192}$ does not satisfy $\Delta_c(x) \geq 0$ near the origin.

If $c < \dfrac{1}{192}$, then $(c - \dfrac{1}{192}) < 0$, so

$$\Delta_c(x) < 0 \qquad \text{for all sufficiently small } x > 0,$$

and therefore no such c is admissible. Thus $\dfrac{1}{196}$, being smaller than $\dfrac{1}{192}$, is the largest constant strictly below the critical threshold $\dfrac{1}{192}$ which guarantees (6). This completes the proof.

Lower bound. A direct computation shows that

$$H''(x) > 0 \qquad \text{for all } x \in (0, \pi/2],$$

so H is strictly convex on $(0, \pi/2]$. Consequently, H admits quadratic supporting functions from below on any subinterval. In particular, there exists a unique quadratic polynomial with maximal quadratic coefficient that supports H from below on $[0, \pi/2]$.

To obtain a sharper bound, we exploit the stronger curvature of H on the right-hand side of the interval and partition $[0, \pi/2]$ into $[0, \pi/3]$ and $[\pi/3, \pi/2]$.

Case I. Verification of $L_1(x) \leq H(x)$ on $[0, \pi/3]$.

Define

$$L_1(x) := \frac{4}{\pi^2}(2 - \sqrt{2}) + \frac{2 - \sqrt{2}}{\pi^4}\, x^2.$$

The polynomial L_1 coincides with the unique quadratic support of H at the endpoint $x = \pi/2$, satisfying

$$L_1\left(\frac{\pi}{2}\right) = H\left(\frac{\pi}{2}\right), \qquad L_1'\left(\frac{\pi}{2}\right) = H'\left(\frac{\pi}{2}\right).$$

By convexity, the graph of H lies above this supporting quadratic on the whole interval $[0, \pi/2]$, and therefore on $[0, \pi/3]$. Optimality follows immediately; since

$$\frac{4}{\pi^2}(2 - \sqrt{2}) + c\, x^2 \leq H(x) \quad \text{on } [0, \pi/2],$$

then evaluating at $x = \pi/2$ yields $c \leq \dfrac{2 - \sqrt{2}}{\pi^4}$.

Case II. Verification of $L_2(x) \leq 2 - \dfrac{\sin(x)}{\sin(x/2)}$ on $[\pi/3, \pi/2]$.

We consider the function

$$g(x) = 2 - \frac{\sin x}{\sin(x/2)}, \qquad x \in [\pi/3, \pi/2].$$

The goal is to construct a quadratic lower bound on the interval $[\pi/3, \pi/2]$ that satisfies

$$L_2(x) \le g(x), \qquad \forall x \in [\pi/3, \pi/2],$$

and is as tight as possible.

We take a quadratic in Taylor-like form around the left endpoint $x_0 = \pi/3$:

$$L_2(x) = g(x_0) + g'(x_0)(x - x_0) + \gamma(x - x_0)^2,$$

where γ is to be determined so that $L(x) \le g(x)$ for all $x \in [\pi/3, \pi/2]$.

Computing values at the left endpoint

$$g(x_0) = g\left(\frac{\pi}{3}\right) = 2 - \frac{\sin(\pi/3)}{\sin(\pi/6)} = 2 - \frac{\sqrt{3}/2}{1/2} = 2 - \sqrt{3}.$$

Next, we compute the derivative of $g(x)$:

$$g'(x) = -\frac{d}{dx}\left(\frac{\sin x}{\sin(x/2)}\right) = -\frac{\cos x \cdot \sin(x/2) - \dfrac{1}{2}\sin x \cdot \cos(x/2)}{\sin^2(x/2)}.$$

Evaluating at $x_0 = \pi/3$:

$$g'(\pi/3) = -\frac{\cos(\pi/3) \cdot \sin(\pi/6) - \frac{1}{2}\sin(\pi/3) \cdot \cos(\pi/6)}{\sin^2(\pi/6)} = \frac{1}{2}.$$

Thus, the quadratic takes the form

$$L_2(x) = (2 - \sqrt{3}) + \frac{1}{2}(x - \pi/3) + \gamma(x - \pi/3)^2.$$

To determine the optimal γ to remain below $g(x)$, we require $p(x_1) \le g(x_1)$ at the right endpoint $x_1 = \pi/2$:

$$L_2(\pi/2) = g(x_0) + g'(x_0)(\pi/2 - \pi/3) + \gamma(\pi/2 - \pi/3)^2 \le g(\pi/2),$$

where

$$g(\pi/2) = 2 - \frac{\sin(\pi/2)}{\sin(\pi/4)} = 2 - \frac{1}{\sqrt{2}/2} = 2 - \sqrt{2}.$$

Compute differences:

$$\pi/2 - \pi/3 = \pi/6, \qquad g(x_1) - g(x_0) = (2 - \sqrt{2}) - (2 - \sqrt{3}) = \sqrt{3} - \sqrt{2}.$$

Thus,

$$(1/2) \cdot \frac{\pi}{6} + \gamma\left(\frac{\pi}{6}\right)^2 \le \sqrt{3} - \sqrt{2}.$$

Solve for γ:

$$\gamma \le \frac{\sqrt{3} - \sqrt{2} - \pi/12}{(\pi/6)^2} = \frac{36}{\pi^2}\left(\sqrt{3} - \sqrt{2} - \frac{\pi}{12}\right).$$

Choosing equality yields the tightest bound. By construction, $L_2(x) \le g(x)$ on the entire interval, and the bound is tangent to $g(x)$ at $x_0 = \pi/3$ and touches the function at $x_1 = \pi/2$, making it tight and smooth.

Optimality of the constant. The quadratic coefficient

$$\gamma = \frac{36}{\pi^2}\Big(\sqrt{3} - \sqrt{2} - \frac{\pi}{12}\Big)$$

is best possible. Indeed, suppose that there exists $\gamma' > \gamma$ such that

$$(2-\sqrt{3}) + \frac{1}{2}\Big(x - \frac{\pi}{3}\Big) + \gamma'(x - \frac{\pi}{3})^2 \;\le\; H(x) \quad \text{for all } x \in [\pi/3, \pi/2].$$

Evaluating this inequality at $x = \pi/2$ yields

$$(2-\sqrt{3}) + \frac{\pi}{12} + \gamma'\Big(\frac{\pi}{6}\Big)^2 \;\le\; 2 - \sqrt{2},$$

which implies $\gamma' \le \gamma$, a contradiction. Hence γ is optimal.

Combining **Cases I** and **II** completes the proof. □

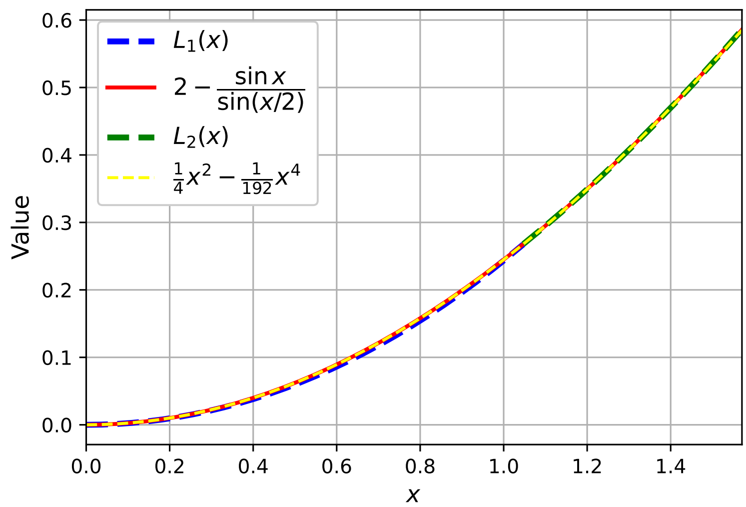

Figure 2. Comparison of the lower bound (blue, green), the function $2 - \frac{\sin(x)}{\sin(x/2)}$ (red), and the upper bound (yellow) over $(0, \pi/2)$. The plot illustrates the tightness of the two rational–radical bounds.

Remark 3.2. *The form of the quadratic L_2 can be heuristically motivated by considering a quadratic ansatz centered at $x = \pi/3$ and maximizing its curvature under the constraint that it remains below the function at $x = \pi/2$. However, the rigorous justification relies on convexity and second-derivative bounds, as presented in Case II, see Figure 2.*

4. New bounds: analytical and graphical proofs

Theorem 4.1. *For all $x \in [0, \pi/2]$, the following refined inequality holds:*

$$\frac{1}{4} - \frac{1}{192}x^2 \le \frac{2 - \dfrac{\sin x}{\sin(x/2)}}{x^2} \le \frac{1}{4} - \frac{1}{196}x^2.$$

The constants $\dfrac{1}{92}$ and $\dfrac{1}{196}$ are best possible.

Proof. Let

$$F(x) = \frac{2 - \dfrac{\sin x}{\sin(x/2)}}{x^2}, \qquad x \in \left(0, \frac{\pi}{2}\right],$$

and define the difference functions

$$D_1(x) = F(x) - \left(\frac{1}{4} - \frac{x^2}{192}\right), \qquad D_2(x) = \left(\frac{1}{4} - \frac{x^2}{196}\right) - F(x).$$

The graphs of D_1 and D_2 show that both functions are nonnegative on $(0, \pi/2]$ and vanish only at $x = 0$. Consequently,

$$\frac{1}{4} - \frac{x^2}{192} \le F(x) \le \frac{1}{4} - \frac{x^2}{196}, \qquad x \in [0, \pi/2].$$

Equivalently, the graph of $F(x)$ is entirely confined between the two quadratic curves, touching them only at the origin. The sharpness of the constant $1/192$ follows from the Taylor expansion

$$F(x) = \frac{1}{4} - \frac{x^2}{192} + O(x^4) \quad (x \to 0),$$

while any improvement of the coefficient $1/196$ would force an intersection between $F(x)$ and the upper quadratic bound inside $(0, \pi/2]$, as confirmed by the plotted graphs. $\square$

To illustrate the sharpness of the refined inequality in Theorem 3.1, define the error functions

$$E^{\text{Lower}}(x) = \left(\frac{1}{4} - \frac{x^2}{192}\right) - \frac{2 - \dfrac{\sin x}{\sin(x/2)}}{x^2},$$

$$E^{\text{Upper}}(x) = \frac{2 - \dfrac{\sin x}{\sin(x/2)}}{x^2} - \left(\frac{1}{4} - \frac{x^2}{196}\right),$$

for $x \in (0, \pi/2]$.

A numerical evaluation shows that both error functions attain their maximum absolute values at the endpoint $x = \pi/2$, and

$$\max_{x \in (0,\pi/2]} \left|E^{\text{Lower}}(x)\right| \approx 1.19 \times 10^{-5},$$

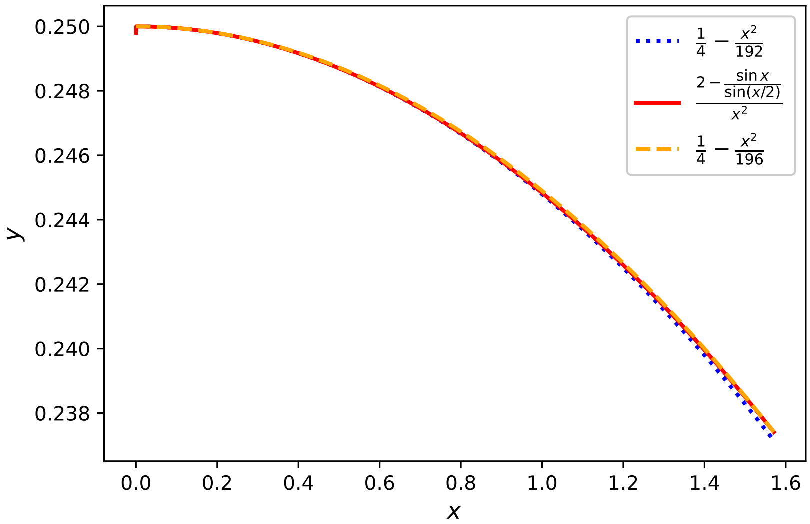

FIGURE 3. Graphs of $F(x) = \frac{2 - \frac{\sin x}{\sin \frac{x}{2}}}{x^2}$ together with the quadratic bounds $\frac{1}{4} - \frac{x^2}{192}$ and $\frac{1}{4} - \frac{x^2}{196}$ on $[0, \pi/2]$. The graph of F remains strictly between the two bounds and touches them only at $x = 0$.

$$\max_{x \in (0,\pi/2]} \left| E^{\text{Upper}}(x) \right| \approx 1.18 \times 10^{-5}.$$

These small error bounds confirm the near–optimality of both the lower and upper estimates on the entire interval $(0, \pi/2]$, see Figure 3.

Theorem 4.2. *For all $x \in (0, \pi/2]$, the inequality*

$$\frac{1}{4} - \frac{1}{192}x^2 \le \frac{2 - \dfrac{\sin x}{\sin(x/2)}}{x^2} \le \frac{1}{4} - \frac{1}{196}x^2.$$

holds.

Proof Without Words. Figures 4 and 5 depict the graphs of the difference functions

$$D_1(x) = F(x) - \left(\frac{1}{4} - \frac{x^2}{192} \right)$$

and

$$D_2(x) = \left(\frac{1}{4} - \frac{x^2}{196} \right) - F(x),$$

for $x \in (0, \pi/2]$.

Both curves remain nonnegative throughout the interval, providing a clear and compelling confirmation of the sharp left and right-hand sides of the inequality–a proof without words.

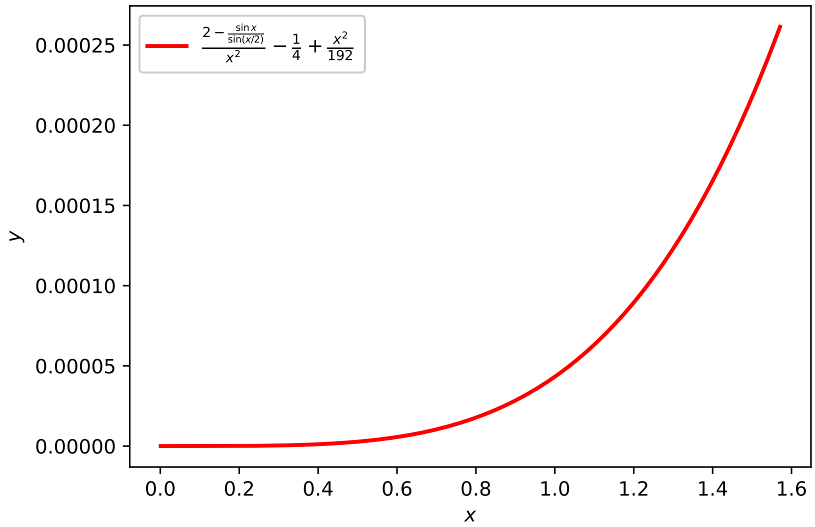

FIGURE 4. Graphs of the difference functions $D_1(x) = F(x) - \left(\frac{1}{4} - \frac{x^2}{192}\right)$ Both curves are nonnegative, confirming the left-hand side inequality.

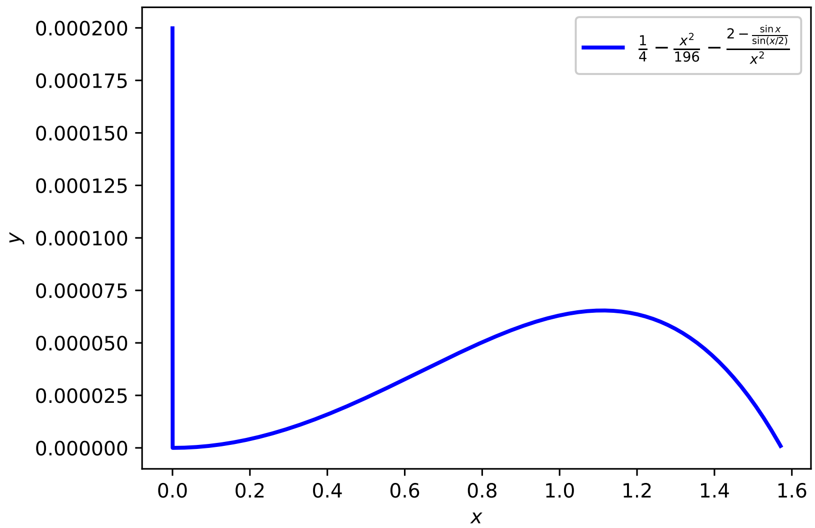

FIGURE 5. Graphs of the difference functions $D_2(x) = \left(\frac{1}{4} - \frac{x^2}{196}\right) - F(x)$ on $(0, \pi/2]$. Both curves are nonnegative, confirming the right-hand side inequality.

Remark 4.1. *Since all the functions involved in the above inequalities are even, all the results can be extended to the interval* $\left[-\frac{\pi}{2}, 0\right)$.

5. **Conclusions**

In this paper, we obtained new polynomial bounds for trigonometric quotients appearing in the D'Aurizio–Sándor inequalities on $(0, \pi/2)$ and derived quadratic and quartic bounds with sharp constants, using convexity properties of suitable normalized functions.

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