ITERATIVE APPROACH TO FRACTIONAL BOUNDARY VALUE PROBLEMS WITH APPLICATIONS

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The aim of this paper is to approximate solutions for a class of boundary value problems involving Caputo fractional-order differential equations by employing a novel iterative scheme. A numerical experiment is presented to illustrate the applicability of the method and to support the theoretical findings. The results obtained extend and unify several well-known comparable results available in the existing literature.

Keywords: iterative algorithm, boundary value problem, convergence, continuous dependence

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1. Introduction

Fractional boundary value problems represent a fundamental area of research and play a key role in developing methods for approximating solutions of fractional differential equations. Researchers are actively engaged in finding, approximating, and characterizing solutions to such problems. The study of fractional boundary value problems typically addresses the existence and uniqueness of solutions, as well as the development of numerical methods to solve them. This also involves exploring different types of fractional derivatives, such as the Riemann–Liouville and Caputo derivatives, in order to establish existence results. For further details in this direction, see [5, 10, 14, 17].

The theory of fixed points provides one of the most powerful tools for establishing existence and uniqueness results in fractional boundary value problems. Many fundamental approaches rely on Banach's contraction principle, Schauder's theorem, and their generalizations to demonstrate that a suitable operator associated with the fractional problem admits one or more fixed points. Such techniques allow researchers to deal not only with classical Caputo

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and Riemann–Liouville derivatives, but also with more recent formulations like the Atangana–Baleanu derivative, which has received increasing attention due to its non-singular kernel and applicability in modeling memory effects [15].

Recent studies highlight the broad applicability of fixed point results in the context of fractional and iterative boundary value problems. For example, Nastasi and Vetro employed fixed point theory to establish existence and uniqueness for first-order periodic differential problems, demonstrating the versatility of such methods beyond the classical fractional setting [8]. Similar approaches have been extended to time-fractional pseudo-parabolic-type equations, where the interplay between nonlinearities and fractional operators requires refined analytical tools [2]. On the numerical side, the combination of fixed point arguments with approximation schemes has led to efficient methods for solving fractional boundary value problems involving Caputo–Fabrizio derivatives [1]. Moreover, fixed point techniques have been successfully applied to iterative and generalized Caputo fractional equations, yielding new results on existence, uniqueness, and convergence of solutions under various boundary conditions [6, 12].

Motivated by the above developments, this paper contributes to the growing literature on fractional boundary value problems by introducing an iterative scheme recently proposed in the literature [4, 13] for approximating solutions of Caputo fractional-order differential equations. Unlike previously studied approaches, the adopted iterative process provides a unified framework that addresses existence and uniqueness for a certain class of mappings. In addition, a numerical example is presented to illustrate the efficiency and applicability of the iterative procedure. The results obtained not only extend and complement several well-known contributions in the field, but also demonstrate the potential of fixed point techniques as a powerful and versatile tool in the analysis of fractional differential equations.

2. **Preliminaries**

Consider the fractional boundary value problem as:

$$\begin{cases} D_a^{\xi} x(t) = \mathcal{G}\left(t, x(t)\right), \ t \in J = [a,b], \ n-1 < \xi < n, \ n \in \mathbb{N} \\ x^{(k)}(a) = c_k, \quad x^{(n-1)}(b) = c_b, \end{cases} \tag{1}$$

for all $k \in \{0, 1, 2, ..., n-2\}$, where D_a^{ξ} denotes the Caputo fractional derivative, $c_k,\ c_b$ are real constants and n is an integer. The function $\mathcal{G} : J \times \mathbb{R} \to \mathbb{R}$ is continuous.

We begin by recalling several definitions and hypotheses that will be needed in the subsequent discussion. Let X be a Banach space with $\|\cdot\|$ and $J = [a, b]$ be an interval of real numbers. Furthermore, we consider $K = \mathbf{C}(J, X)$, which denotes a Banach space of all continuous functions from J to X, equipped with the norm $\|x\|_B = \sup\{\|x(t)\| \ : \ t \in J\}$.

Throughout the paper, we assume that there exists a function $k \in \mathbf{C}(J, \mathbb{R}^+)$ such that

$$\|\mathcal{G}(t,x) - \mathcal{G}(t,y)\| \leq k(t)\|x - y\|, \tag{2}$$

for all $t \in J$ and for all $x, y \in X$.

Definition 2.1. *Let $x \in \mathbf{C}^1[a,b]$. The left-sided Riemann-Liouville fractional integral of order $\xi \in (0,\infty)$ is defined as*

$$I_a^\xi x(t) = \frac{1}{\Gamma(\xi)} \int_a^t (t-s)^{\xi-1} x(s) ds, \quad t \in [a,b],$$

where Γ is the Euler gamma function.

Definition 2.2. *Let $x \in \mathbf{C}^n[a,b]$ and let $n-1 < \xi \leq n$, where $n \in \mathbb{N}$. The left-sided Riemann–Liouville fractional derivative of order ξ of x is defined by*

$$D_a^\xi x(t) = \frac{d^n}{dt^n}\left[I_a^{n-\xi} x(t)\right] = \frac{1}{\Gamma(n-\xi)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\xi-1} x(s)\, ds, \quad t \in [a,b].$$

Definition 2.3. *Let $x \in \mathbf{C}^n[a,b]$ and let $n-1 < \xi \leq n$, where $n \in \mathbb{N}$. The left-sided Caputo fractional derivative of order ξ of x is defined by*

$$D_a^\xi x(t) = I_a^{n-\xi} x^{(n)}(t) = \frac{1}{\Gamma(n-\xi)} \int_a^t (t-s)^{n-\xi-1} x^{(n)}(s)\, ds, \quad t \in [a,b].$$

Lemma 2.1 ([14])**.** *A function $x \in K$ is a solution of the fractional boundary value problem defined in (1) if and only if $x(t)$ is a solutions of the fractional integral equation*

$$\begin{aligned} x(t) = &\sum_{k=0}^{n-2} \frac{c_k}{k!}(t-a)^k + \left(\frac{c_b}{(n-1)!} + \frac{\mathcal{G}(a,x(a))(b-a)^{\xi-n+1}}{(n-2)!\Gamma(\xi-n+2)}\right)(t-a)^{n-1} \\ &- \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\xi-n+1)} \int_a^b (b-s)^{\xi-n} \mathcal{G}(s,x(s)) ds \\ &+ \frac{1}{\Gamma(\xi)} \int_a^t (t-s)^{\xi-1} \mathcal{G}(s,x(s)) ds. \end{aligned}$$

Lemma 2.2 ([16])**.** *Let $\{\xi_n\}$ and $\{\eta_n\}$ be nonnegative sequences for which the following inequality holds*

$$\xi_{n+1} \leq (1-\mu_n)\xi_n + \mu_n \eta_n,$$

where $\mu_n \in (0,1)$, for all $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty} \mu_n = \infty$, then we have

$$0 \leq \limsup_{n\to\infty} \xi_n \leq \limsup_{n\to\infty} \eta_n.$$

In recent years, researchers have developed numerous significant iterative algorithms to approximate the solutions of nonlinear operators. One of the simplest and most fundamental iterative methods was introduced by Banach and is known as the Picard iteration process [9]. Given an initial point x_0 in a closed subset K of a complete metric space, the Picard sequence $\{x_n\}$ is defined by

$$x_{n+1} = Tx_n,$$

for $n \in \mathbb{N}$, where $T : K \to K$ is a self-mapping.

It is well known that the iterative sequence may fail to converge to the fixed point of T if, in the above process, the contraction mapping T is replaced by a nonexpansive mapping. To address this issue, Mann [7] introduced a foundational iterative scheme for nonexpansive mappings, defined as follows: Let x_0 be an arbitrary point in a closed and convex subset K of a normed space X. Define the sequence $\{x_n\}$ in K by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is an appropriate real sequence in $[0, 1]$. The Mann iterative algorithm may fail to converge when a nonexpansive mapping is replaced by a Lipschitz pseudocontractive mapping. To overcome this limitation, Ishikawa [3] introduced a two-step iterative scheme, defined as follows: let x_0 be an arbitrary point in a closed and convex subset K of a normed space X. Define a sequence $\{x_n\}$ in K by

$$\begin{cases} y_n = (1 - \alpha_n)x_n + \alpha_n Tx_n \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n Ty_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$. Sahu [11] introduced the S-iteration process in order to improve the convergence properties of the Ishikawa iterative scheme. This iterative process is defined as follows: let x_0 be an arbitrary point in a closed and convex subset K of a normed space X. Define a three-step iterative sequence $\{x_n\}$ in K by

$$\begin{cases} y_n = (1 - \alpha_n)x_n + \alpha_n Tx_n \\ x_{n+1} = (1 - \beta_n)Tx_n + \beta_n Ty_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$.

We now turn our attention to a new iterative process, which was recently introduced in [13]. Let T be a mapping defined on a nonempty subset K of a Banach space and $x_0 \in K$, an arbitrary point. Define a sequence $\{x_n\}$ by

$$\begin{cases} w_n = T\left[(1 - \alpha_n)x_n + \alpha_n Tx_n\right] \\ z_n = T\left[(1 - \beta_n)w_n + \beta_n Tw_n\right] \\ y_n = T\left[(1 - \gamma_n)z_n + \gamma_n Tz_n\right] \\ x_{n+1} = Ty_n, \end{cases} \tag{3}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are real sequences in $[0, 1]$. For simplicity, this process will be referred to as the PJ iteration.

Further details regarding the efficiency of the iterative process (such as achieving the fixed point value in the minimal number of steps, minimizing the error between the exact and numerical solution, and reducing CPU time) can be found in [4].

The aim of this paper is to establish the existence and uniqueness of solutions to a fractional boundary value problem by employing the new iterative scheme. Finally, we present a numerical experiment to validate the effectiveness of our proposed technique.

3. **Convergence analysis**

Define the operator T on Banach space K for the problem (1) as

$$
\begin{aligned}
(Tx)(t) = &\sum_{k=0}^{n-2} \frac{c_k}{k!}(t-a)^k + \left(\frac{c_b}{(n-1)!} + \frac{\mathcal{G}(a,x(a))(b-a)^{\xi-n+1}}{(n-2)!\Gamma(\xi-n+2)}\right)(t-a)^{n-1} \\
&- \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\xi-n+1)} \int_a^b (b-s)^{\xi-n}\mathcal{G}(s,x(s))ds \\
&+ \frac{1}{\Gamma(\xi)} \int_a^t (t-s)^{\xi-1}\mathcal{G}(s,x(s))ds.
\end{aligned} \tag{4}
$$

The following theorem gives the existence and uniqueness of solutions of fractional boundary value problem (1).

Theorem 3.1. *Suppose that $\mathcal{G}(t,x(t))$ satisfies condition (2) and T is defined as in (4). Suppose $\{x_n\}$ is a sequence defined in (3) with $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then, $\{x_n\}$ converges to the solution of the boundary value problem (1) in K, if*

$$
\Phi = \left[\frac{(b-a)^{\xi}}{(n-2)!\Gamma(\xi-n+2)}k(a) + \frac{(b-a)^{n-1}}{(n-1)!}I_a^{\xi-n+1}k(b) + I_a^{\xi}k(t)\right] < 1.
$$

Proof. We show that the sequence $\{x_n\}$ converges to x for the operator T. Using the given assumptions, we obtain

$$
\begin{aligned}
\|x_{n+1}(t) - x(t)\| &= \|(Ty_n)(t) - (Tx)(t)\| \\
&= \| \sum_{k=0}^{n-2} \frac{c_k}{k!}(t-a)^k + \left(\frac{c_b}{(n-1)!} + \frac{\mathcal{G}(a,y_n(b))(b-a)^{\xi-n+1}}{(n-2)!\Gamma(\xi-n+2)}\right)(t-a)^{n-1} \\
&- \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\xi-n+1)} \int_a^b (b-s)^{\xi-n}\mathcal{G}(s,y_n(s))\,ds \\
&+ \frac{1}{\Gamma(\xi)} \int_a^t (t-s)^{\xi-1}\mathcal{G}(s,y_n(s))\,ds
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{k=0}^{n-2} \frac{c_k}{k!}(t-a)^k - \left(\frac{c_b}{(n-1)!} + \frac{\mathcal{G}(a, x(b))(b-a)^{\xi-n+1}}{(n-2)!\Gamma(\xi-n+2)} \right)(t-a)^{n-1} \\
&+ \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\xi-n+1)} \int_a^b (b-s)^{\xi-n} \mathcal{G}(s, x(s))ds \\
&- \frac{1}{\Gamma(\xi)} \int_a^t (t-s)^{\xi-1} \mathcal{G}(s, x(s))ds\| \\
&\leq \left(\frac{\left\| \mathcal{G}\left(a, y_n(b)\right) - \mathcal{G}(a, x(b)) \right\| (b-a)^{\xi-n+1}}{(n-2)!\Gamma(\xi-n+2)} \right)(t-a)^{n-1} \\
&+ \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\xi-n+1)} \int_a^b (b-s)^{\xi-n} \left\| \mathcal{G}\left(s, y_n(s)\right) - \mathcal{G}(s, x(s)) \right\| ds \\
&+ \frac{1}{\Gamma(\xi)} \int_a^t (t-s)^{\xi-1} \left\| \mathcal{G}\left(s, y_n(s)\right) - \mathcal{G}(s, x(s)) \right\| ds \\
&\leq \left(\frac{k(a) \left\| y_n(b) - x(b) \right\| (b-a)^{\xi-n+1}}{(n-2)!\Gamma(\xi-n+2)} \right)(t-a)^{n-1} \\
&+ \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\xi-n+1)} \int_a^b (b-s)^{\xi-n} k(s) \left\| y_n(s) - x(s) \right\| ds \\
&+ \frac{1}{\Gamma(\xi)} \int_a^t (t-s)^{\xi-1} k(s) \left\| y_n(s) - x(s) \right\| ds.
\end{aligned}
$$

Now, by taking the supremum in the last inequality, we have

$$
\begin{aligned}
\|x_{n+1} - x\|_B &\leq \left(\frac{k(a)(t-a)^{n-1}(b-a)^{\xi-n+1}}{(n-2)!\Gamma(\xi-n+2)} \right) \|y_n - x\|_B \\
&+ \frac{\|y_n - x\|_B (t-a)^{n-1}}{(n-1)!\Gamma(\xi-n+1)} \int_a^b (b-s)^{\xi-n} k(s)ds \\
&+ \frac{\|y_n - x\|_B}{\Gamma(\xi)} \int_a^t (t-s)^{\xi-1} k(s)ds \\
&\leq \left(\frac{k(a)(t-a)^{n-1}(b-a)^{\xi-n+1}}{(n-2)!\Gamma(\xi-n+2)} \right) \|y_n - x\|_B \\
&+ \left[\frac{(t-a)^{n-1}}{(n-1)!} I_a^{\xi-n+1} k(b) + I_a^{\xi} k(t) \right] \|y_n - x\|_B \\
&\leq \left(\frac{k(a)(b-a)^{n-1}(b-a)^{\xi-n+1}}{(n-2)!\Gamma(\xi-n+2)} \right) \|y_n - x\|_B \\
&+ \left[\frac{(b-a)^{n-1}}{(n-1)!} I_a^{\xi-n+1} k(b) + I_a^{\xi} k(t) \right] \|y_n - x\|_B \\
&\leq \left[\frac{(b-a)^{\xi} k(a)}{(n-2)!\Gamma(\xi-n+2)} + \frac{(b-a)^{n-1}}{(n-1)!} I_a^{\xi-n+1} k(b) + I_a^{\xi} k(t) \right] \|y_n - x\|_B,
\end{aligned}
$$

leading to $\|x_{n+1}-x\|_B \leq \Phi \|y_n - x\|_B$. Hence, we conclude that

$$\|x_{n+1}-x\|_B \leq \Phi\|y_n - x\|_B. \tag{5}$$

On the other hand,

$$\begin{aligned}
\|w_n - x\|_B &= \|T((1-\alpha_n)x_n + \alpha_n Tx_n) - Tx\|_B \\
&\leq \Phi \|(1-\alpha_n)x_n + \alpha_n Tx_n - x\|_B \\
&\leq \Phi\{(1-\alpha_n)\|x_n - x\|_B + \alpha_n \|Tx_n - Tx\|_B\} \\
&\leq \Phi\{(1-\alpha_n)\|x_n - x\|_B + \alpha_n \Phi \|x_n - x\|_B\} \\
&= \Phi\{(1-\alpha_n(1-\Phi))\} \|x_n - x\|_B,
\end{aligned}$$

$$\begin{aligned}
\|z_n - x\|_B &= \|T((1-\beta_n)w_n + \beta_n Tw_n) - Tx\|_B \\
&\leq \Phi \|(1-\beta_n)w_n + \beta_n Tw_n - x\|_B \\
&\leq \Phi (1-\beta_n)\|w_n - x\|_B + \beta_n \|Tw_n - Tx\|_B \\
&\leq \Phi (1-\beta_n)\|w_n - x\|_B + \beta_n \Phi \|w_n - x\|_B\} \\
&= \Phi \|w_n - x\|_B \\
&\leq \Phi^2\{(1-\alpha_n(1-\Phi))\} \|x_n - x\|_B,
\end{aligned}$$

and

$$\begin{aligned}
\|y_n - x\|_B &= \|T((1-\gamma_n)z_n + \gamma_n Tz_n) - Tx\|_B \\
&\leq \Phi \|(1-\gamma_n)z_n + \gamma_n Tz_n - x\|_B \\
&\leq \Phi (1-\gamma_n)\|z_n - x\|_B + \gamma_n \|Tz_n - Tx\|_B \\
&\leq \Phi (1-\gamma_n)\|z_n - x\|_B + \gamma_n \Phi \|z_n - x\|_B\} \\
&= \Phi \|z_n - x\|_B \\
&\leq \Phi^3\{(1-\alpha_n(1-\Phi))\} \|x_n - x\|_B.
\end{aligned}$$

It follows from (5) that

$$\begin{aligned}
\|x_{n+1} - x\|_B &\leq \Phi \|y_n - x\|_B \\
&\leq \Phi^4 (1-\alpha_n(1-\Phi)) \|x_n - x\|_B.
\end{aligned}$$

Using $\Phi \in (0,1)$ and $(1-\alpha_n(1-\Phi)) < 1$, we obtain

$$\begin{cases}
\|x_{n+1} - x\|_B \leq \Phi^4 (1-\alpha_n(1-\Phi)) \|x_n - x\|_B \\
\|x_n - x\|_B \leq \Phi^4 (1-\alpha_{n-1}(1-\Phi)) \|x_{n-1} - x\|_B \\
\|x_{n-1} - x\|_B \leq \Phi^4 (1-\alpha_{n-2}(1-\Phi)) \|x_{n-2} - x\|_B \\
\vdots \\
\|x_1 - x\|_B \leq \Phi^4 (1-\alpha_0(1-\Phi)) \|x_0 - x\|_B.
\end{cases}$$

This leads to

$$\|x_{n+1} - x\|_B \leq \Phi^{4n} \|x_0 - x\|_B \prod_{i=0}^{n} (1 - \alpha_i(1 - \Phi)).$$

Using the facts that $(1 - \alpha_m(1 - \Phi)) < 1$ and $1 - r \leq e^{-r}$, for all $r \in (0, 1)$, we have

$$\|x_{n+1} - x\|_B \leq \frac{\Phi^{4n} \|x_0 - x\|_B}{e^{(1-\Phi)\sum_{i=0}^{n} \alpha_i}}, \tag{6}$$

This implies that $\lim_{n\to\infty} \|x_n - x\|_B = 0$. Hence, $\{x_n\}$ converges to x. $\square$

Next, we will make two important observations regarding the result obtained previously, which will shed light on its underlying structure and help clarify its implications for the subsequent analysis.

Remark 3.1. *It is noted that inequality (6) provides a bound in terms of known functions, so it can be explicitly evaluated and used to estimate the behavior of solutions for the fractional boundary value problem (1).*

Remark 3.2. *The fixed point obtained in the previous theorem belongs to the Banach space $C(J, \mathbb{R})$. However, since the solution satisfies the equivalent integral equation given in Lemma 2.1 and the function $\mathcal{G}$ is continuous, it follows from the properties of fractional integral operators that the solution x possesses higher regularity. In particular, we have $x \in C^{n-1}(J, \mathbb{R})$, which denotes the space of all $(n-1)$-times continuously differentiable functions from J to $\mathbb{R}$.*

4. **A result on the continuous dependence of solutions**

We begin by presenting several elements that are necessary for establishing the main result of this section.

Definition 4.1. *Let T and $\tilde{T}$ be two operators defined on K. Then $\tilde{T}$ is called an approximate operator of T if there exists $\epsilon > 0$ such that*

$$\|Tx - \tilde{T}x\|_B \leq \epsilon,$$

for all $x \in K$.

We consider the iterative sequence $\tilde{x}_n$ for the operator $\tilde{T}$ as

$$\begin{cases} \tilde{w}_n = \tilde{T}\big[(1-\alpha_n)\tilde{x}_n + \alpha_n \tilde{T}\tilde{x}_n\big] \\ \tilde{z}_n = \tilde{T}\big[(1-\beta_n)\tilde{w}_n + \beta_n \tilde{T}\tilde{w}_n\big] \\ \tilde{y}_n = \tilde{T}\big[(1-\gamma_n)\tilde{z}_n + \gamma_n \tilde{T}\tilde{z}_n\big] \\ \tilde{x}_{n+1} = \tilde{T}\tilde{y}_n, \end{cases} \tag{7}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1]$. Suppose $x(t)$ and $\tilde{x}(t)$ denote the solutions of the boundary value problem (1) with the boundary conditions

$$\begin{cases} x^{(k)}(a) = c_k, & x^{(n-1)}(b) = c_b \\ \tilde{x}^{(k)}(a) = \tilde{c}_k, & \tilde{x}^{(n-1)}(b) = \tilde{c}_b, \end{cases}$$

for all $k \in \{0, 1, 2, ..., n-2\}$.

The following theorem addresses the continuous dependence of the solutions of the fractional boundary value problem (1) on the boundary data.

Theorem 4.1. *Let $T, \tilde{T}$ be operators defined on K as given in (4). Suppose $\{x_n\}$ and $\{\tilde{x}_n\}$ are sequences defined by (3) and (7), associated with T and $\tilde{T}$, respectively, satisfying $\beta_n\gamma_n \geq \frac{1}{2}$ for all $n \in \mathbb{N}$. If $\lim_{n\to\infty} \tilde{x}_n = \tilde{x}$, then we have*

$$\|x - \tilde{x}\|_B \leq \frac{13M}{1-\Phi},$$

where

$$M = \sum_{k=0}^{n-2} \frac{\|c_k - \tilde{c}_k\|}{k!}(b-a)^k + \frac{\|c_b - \tilde{c}_b\|}{(n-1)!}(b-a)^{n-1}.$$

Sketch of the proof. By comparing the operators T and $\tilde{T}$ and using the Lipschitz condition on $\mathcal{G}$, one obtains

$$\|x_{n+1} - \tilde{x}_{n+1}\|_B \leq M + \Phi\|y_n - \tilde{y}_n\|_B,$$

with $\Phi < 1$. Propagating this estimate through the iterative scheme yields

$$\|x_{n+1} - \tilde{x}_{n+1}\|_B \leq (1-\mu_n)\|x_n - \tilde{x}_n\|_B + \mu_n \frac{13M}{1-\Phi},$$

where $\mu_n = \beta_n\gamma_n(1-\Phi)$.

Since $\beta_n\gamma_n \geq \frac{1}{2}$, we have $\sum_{n=0}^{\infty} \mu_n = \infty$, and by Lemma 2.2 it follows that

$$\|x - \tilde{x}\|_B \leq \frac{13M}{1-\Phi},$$

and this completes the proof. □

We note that we have chosen to present only a sketch of the proof, as the intermediate estimates are technical. Moreover, we emphasize that this result is also important for the practical part, to which we will refer in the next section.

5. **Illustrative applications and comparative analysis**

Let us consider the problem

$$\begin{cases} \left(D_a^\xi\right) x(t) = t\left[\frac{t-\cos(x(t))}{2}\right], \ t \in J = [0,1], \ n-1 < \xi \leq n, \ n \in \mathbb{N} \\ x^{(k)}(0) = 0, \quad x^{(n-1)}(1) = 1, \end{cases}$$

for all $k \in \{0, 1, 2, ..., n-2\}$.

By comparing with the problem (1), we get

$$\mathcal{G}(t,x(t)) = t\left[\frac{t-\cos(x(t))}{2}\right]. \tag{8}$$

5.1. Existence and uniqueness of solution

Using (8), we obtain

$$\begin{aligned}\|\mathcal{G}(t,x(t))-\mathcal{G}(t,\tilde{x}(t))\| &\leq \left|\frac{t}{2}\right|\cdot\|t-\cos(x(t))-t-\cos(\tilde{x}(t))\|\\ &\leq \frac{t}{2}\,\|\cos(x(t))-\cos(\tilde{x}(t))\| \leq \frac{t}{2}\,\|x(t)-\tilde{x}(t)\|,\end{aligned}$$

where $k(t)=\frac{t}{2}$. Now, we estimate

$$\begin{aligned}\Phi &= \frac{(b-a)^{\xi}k(a)}{(n-2)!\Gamma(\xi-n+2)}+\frac{(b-a)^{n-1}}{(n-1)!}I_a^{\xi-n+1}k(b)+I_a^{\xi}k(t)\\ &= \frac{k(0)}{(n-2)!\Gamma(\xi-n+2)}+\frac{1}{(n-1)!}I^{\xi-n+1}k(1)+I^{\xi}k(t)\\ &= \frac{0}{(n-2)!\Gamma(\xi-n+2)}+\frac{1}{(n-1)!}I^{\xi-n+1}k(1)+I^{\xi}k(t)\\ &\leq \frac{1}{2}\left[\frac{1}{(n-1)!\Gamma(\xi-n+1)}\int_0^1(1-s)^{\xi-n}sds+\frac{1}{\Gamma(\xi)}\int_0^t(t-s)^{\xi-1}sds\right]\\ &= \frac{1}{2}\left[\frac{1}{(n-1)!\Gamma(\xi-n+1)}B(2,\xi-n+1)+\frac{1}{\Gamma(\xi)}B(2,\xi)\right]\\ &= \frac{1}{2}\left[\frac{1}{(n-1)!\Gamma(\xi-n+3)}+\frac{1}{\Gamma(\xi+2)}\right].\end{aligned}$$

If $\frac{1}{2}\left[\frac{1}{(n-1)!\Gamma(\xi-n+3)}+\frac{1}{\Gamma(\xi+2)}\right]<1$, then $\Phi<1$.

For instance, let $\xi=\frac{5}{2}$. Then, $n=[\xi]+1=\left[\frac{5}{2}\right]+1=2+1=3$, and

$$\begin{aligned}\Phi &\leq \frac{1}{2}\left[\frac{1}{(3-1)!\Gamma\left(\frac{5}{2}-3+3\right)}+\frac{1}{\Gamma\left(\frac{5}{2}+2\right)}\right]=\frac{1}{2}\left[\frac{1}{2\Gamma\left(\frac{5}{2}\right)}+\frac{1}{\Gamma\left(\frac{9}{2}\right)}\right]\\ &= \frac{1}{\sqrt{\pi}}\left(\frac{1}{3}+\frac{8}{105}\right)=\frac{43}{105\sqrt{\pi}}\simeq 0.2310<1.\end{aligned}$$

We define the operator T on Banach space K for the given problem by

$$\begin{aligned}(Tx)(t)=&\frac{t^2}{2}-\frac{1}{2\sqrt{\pi}}t^2\int_0^1(1-s)^{-\frac{1}{2}}s\left[\frac{s-\cos(x(s))}{2}\right]ds\\ &+\frac{4}{3\sqrt{\pi}}\int_0^t(t-s)^{\frac{1}{2}}s\left[\frac{s-\cos(x(s))}{2}\right]ds,\quad t\in J.\end{aligned} \tag{9}$$

Since all the conditions of Theorem 3.1 are satisfied, it follows from its conclusion that the sequence $\{x_n\}$, generated by the new iterative scheme (3) for the operator T in (9), converges to a unique solution $x \in K$.

5.2. Estimation of the error

For any $x_0 \in K$, we have

$$\begin{aligned} \|x_{n+1} - x\|_B &\leq \frac{\Phi^n}{e^{(1-\Phi)\sum_{i=0}^{n}\alpha_i}} \|x_0 - x\|_B \\ &\leq \frac{\left(\frac{43}{105\sqrt{\pi}}\right)^n}{e^{\left(1-\frac{43}{105\sqrt{\pi}}\right)\sum_{i=0}^{n}\alpha_i}} \|x_0 - x\|_B \\ &\leq \frac{\left(\frac{43}{105\sqrt{\pi}}\right)^n}{e^{\left(1-\frac{43}{105\sqrt{\pi}}\right)\sum_{i=0}^{n}\frac{1}{1+i}}} \|x_0 - x\|_B \,, \end{aligned}$$

where we have chosen $\alpha_i = \dfrac{1}{1+i} \in [0,1]$. The resulting estimate provides an upper bound for the error arising from the truncation in the computation.

5.3. Continuous dependence

Now, we investigate the continuous dependence of the solutions of equation (1) on the boundary data. In particular, for

$$c_0 = c_1 = \tilde{c}_0 = \tilde{c}_1 = 0, \quad c_b = 1, \quad \tilde{c}_b = \frac{3}{4},$$

we obtain

$$\begin{aligned} \|x - \tilde{x}\|_B &\leq \frac{13M}{1-\Phi} \\ &\leq \frac{13\left[\sum_{k=0}^{n-2}\frac{\|c_k - \tilde{c}_k\|}{k!}(b-a)^k + \frac{\|c_b - \tilde{c}_b\|}{(n-1)!}(b-a)^{n-1}\right]}{1-\Phi} \\ &= \frac{13\cdot\frac{1}{8}}{1-\frac{43}{105\sqrt{\pi}}} = \frac{13}{6.1512} \simeq 2.1134. \end{aligned}$$

5.4. Comparison analysis

Now, we discuss the simplicity and efficiency of the new iteration scheme. Referring to [9, 7, 3, 11], we define the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, and $\{e_n\}$ corresponding to the PJ algorithm (3), the Picard iteration, the Mann iteration, the Ishikawa iteration, and the S-iteration, respectively:

$$\begin{aligned} a_n =&\delta^{4n}[1-\alpha(1-\delta)]\|u_0-x\|,\\ b_n =&\delta^{n}\|v_0-x\|,\\ c_n =&[1-\alpha(1-\delta)]^n\|w_0-x\|,\\ d_n =&[1-\alpha(1-\delta^2)]^n\|x_0-x\|,\\ e_n =&\delta^{n}[1-\alpha\beta(1-\delta)]\|y_0-x\|, \end{aligned}$$

where $\delta \in [0,1)$ denotes the contraction constant. For a given initial guess, the convergence of sequences a_n, b_n, c_n, d_n and e_n depends solely on the factors $\Phi_1 = \delta^{4n}[1-\alpha(1-\delta)]^n$, $\Phi_2 = \delta^n, \Phi_3 = [1-\alpha(1-\delta)]^n$, $\Phi_4 = [1-\alpha(1-\delta^2)]^n$ and $\Phi_5 = \delta^n[1-\alpha\beta(1-\delta)]^n$, respectively.

Therefore, Figure 1 and Table 1 present the values of these factors under the corresponding iterative processes for the numerical example discussed in this paper, with $\delta = \Phi = 0.138629441$ and $\xi_n = \beta_n = \dfrac{1}{2}$. Based on these data, it can be observed that the new iteration scheme (3) converges significantly faster to the exact value compared to the Picard iteration, Mann iteration, Ishikawa iteration, and S-iteration.

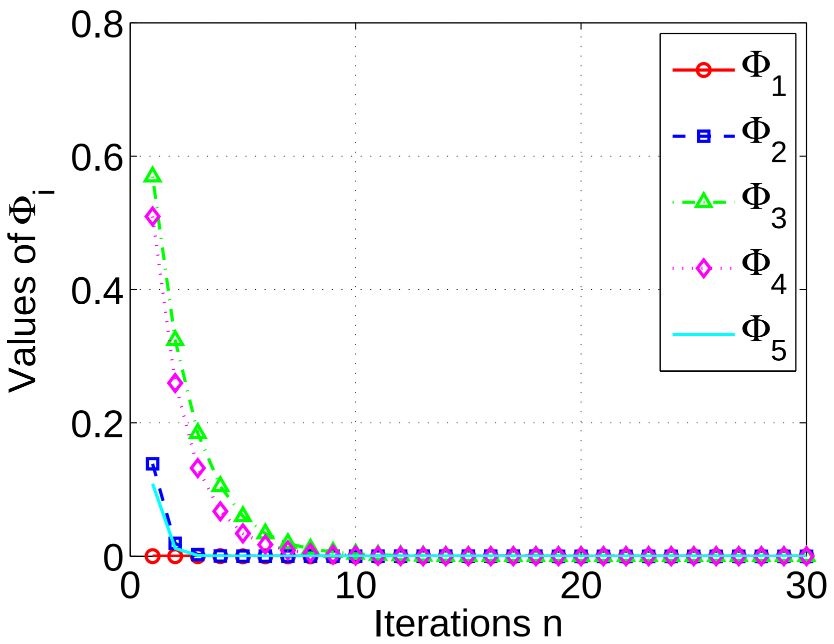

FIGURE 1. Graphical comparison of $\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5$.

TABLE 1. Numerical comparison of $\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5$.

No.	Φ_1 (PJ)	Φ_2 (Picard)	Φ_3 (Mann)	Φ_4 (Ishikawa)	Φ_5 (S-iteration)
1	0.0002102700	0.1386300000	0.5693100000	0.5096100000	0.1087800000
2	0.0000000442	0.0192180000	0.3241200000	0.2597000000	0.0118320000
3	0.0000000000	0.0026642000	0.1845300000	0.1323500000	0.0012871000
4	0.0000000000	0.0003693400	0.1050500000	0.0674450000	0.0001400000
5	0.0000000000	0.0000512010	0.0598080000	0.0343700000	0.0000152290
6	0.0000000000	0.0000070979	0.0340500000	0.0175160000	0.0000016566
7	0.0000000000	0.0000009839	0.0193850000	0.0089261000	0.0000001802
8	0.0000000000	0.0000001364	0.0110360000	0.0045488000	0.0000000196
9	0.0000000000	0.0000000189	0.0062831000	0.0023181000	0.0000000021
10	0.0000000000	0.0000000026	0.0035770000	0.0011813000	0.0000000002
11	0.0000000000	0.0000000004	0.0020365000	0.0006020200	0.0000000000
12	0.0000000000	0.0000000001	0.0011594000	0.0003067900	0.0000000000
13	0.0000000000	0.0000000000	0.0006600600	0.0001563400	0.0000000000
14	0.0000000000	0.0000000000	0.0003757800	0.0000796750	0.0000000000
15	0.0000000000	0.0000000000	0.0002139400	0.0000406030	0.0000000000
16	0.0000000000	0.0000000000	0.0001218000	0.0000206920	0.0000000000
17	0.0000000000	0.0000000000	0.0000693400	0.0000105450	0.0000000000
18	0.0000000000	0.0000000000	0.0000394800	0.0000053736	0.0000000000
19	0.0000000000	0.0000000000	0.0000224800	0.0000027385	0.0000000000
20	0.0000000000	0.0000000000	0.0000128000	0.0000013955	0.0000000000

6. Conclusions

In this research, we established the existence and uniqueness of solutions for a class of fractional boundary value problems by employing a recently introduced iterative scheme. Furthermore, we examined the continuous dependence of the solutions on the given data.

To further validate the effectiveness of the proposed method, we provided a concrete numerical example that illustrates the theoretical results and demonstrates the computational efficiency of the scheme. In particular, a comparison table shows that the proposed iterative process converges faster than several well-known existing methods.

The results presented in this paper extend and complement several known contributions in the literature, thereby enriching the analytical framework for fractional boundary value problems. The proposed methodology has the potential to be extended to broader classes of fractional differential equations and more general boundary conditions. Future work may explore its applicability to systems with variable-order derivatives, nonlocal conditions, or models arising in applied sciences where fractional dynamics play a central role.

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